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Constrained quantum particles and geometric phases in noninertial frames

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Abstract
We consider particles constrained to move in the close vicinity of a space curve through a steep quadratic potential in the plane normal to the curve. As is known, the effective 1D Hamiltonian that governs the motion along the curve involves both its curvature and torsion. Thus, the adiabatic cyclic deformations of the curve might give rise to geometric phases being accumulated by the particle. We report that this is indeed, in general, the case, and analyze in detail the origin of the phases, as described in both an inertial Cartesian frame and a noninertial one, adapted to the curve. We extend previous work to include non-locally arclength preserving deformations, and derive general formulas for Berry’s curvature in this case.

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1. Introduction

In this work we put together the following two well-known facts.

(1) Quantum systems, the Hamiltonians of which depend on external parameters, may accumulate geometric phases as a result of the cyclic, adiabatic change of the parameters (see [1, 17] for the original formulation, [26] for the non-Abelian generalization and, e.g., [4, 18, 19, 21, 22, 27] for experimental verification).

(2) When quantum particles are constrained to move along a space curve through a steep quadratic potential in the plane normal to the curve, the effective 1D Hamiltonian that governs their motion along the curve involves both the curvature and the torsion of the curve (see [7–12, 14, 15, 20] for various aspects of this problem, and higher dimensional generalizations).
As a result, we ponder on the possibility of geometric phases showing up when the above space curve is adiabatically and cyclically deformed. The effect, which we indeed find, allows, in principle, the detection of mechanical deformations, even when the body being deformed returns exactly to its initial configuration.

We study the problem from the point of view of both an inertial observer, using Cartesian coordinates, and a noninertial one, using coordinates adapted to the moving curve. In the latter case, the effective 1D Hamiltonian for motion along the curve depends not only on the instantaneous shape of the curve, but also on its velocity, a feature simulating inertial forces felt in this frame. While in the case of the Cartesian frame we employ the standard approach to calculating Berry’s curvature, in the adapted frame case, we opt for a more direct method, based on time-dependent perturbation theory, the reason being that in this case, part of the geometric phase resides in the dynamical one. In both cases, we derive general formulas for Berry’s curvature and comment on their relation. The fact that we include in our study non-locally arclength preserving deformations complicates the analysis, as the measure used in the Hilbert space inner product between states is nonuniform and time varying.

The structure of the paper is as follows. In sections 2 and 3, we give the basic background material related, respectively, to facts (i) and (ii) above. Section 4 provides the synthesis mentioned, calculating the geometric phase in a Cartesian frame, while in section 5, we approach the problem in the adapted frame, using time-dependent perturbation theory. Section 6 rounds up our analysis by pointing out directions for future work, as well as possible applications.

2. Quantum geometric phases

Consider a physical system described by a Hamiltonian $H$ that depends on a set of parameters $\xi^A$, $H = H(\xi)$. For each value of the parameters, the corresponding time-independent Schrödinger equation

$$H(\xi)|n, \xi\rangle = E^\xi_n |n, \xi\rangle \quad (1)$$

is satisfied by an orthonormal set of eigenvectors $|n, \xi\rangle$. Assume that the system, at $t = 0$, is in a state described by a non-degenerate eigenvector $|n, \xi_0\rangle$, and the $\xi$ are subsequently varied in time, following a curve $C$ in parameter space. If the change in the $\xi$ is slow enough, in the time scale set by the energy difference of neighboring states, and, throughout the experiment, no level crossing involving $|n, \xi_t\rangle$ takes place, the adiabatic theorem (see, e.g., [13], chapter 17, section 13) states that the system locks onto $|n, \xi_t\rangle$, picking up at most a phase,

$$|\psi_t\rangle = e^{-i\int_0^t d\tau E^\xi_\tau} e^{i\gamma^{(n)}_t |n, \xi_t\rangle}, \quad (2)$$

where the first phase factor is the expected dynamical one and $\gamma^{(n)}_t$ is a possible correction. Substitution into the time-dependent Schrödinger equation shows that

$$\gamma^{(n)}_t = i \int_0^t d\tau \langle n, \xi_\tau | \frac{d}{d\tau} | n, \xi_\tau \rangle$$

$$= i \int_C \langle n, \xi | d | n, \xi \rangle \quad (3)$$

where $d \equiv (d\xi^A) \partial / \partial \xi^A$ is the exterior derivative in $\xi$-space—the latter form emphasizes the fact that $\gamma^{(n)}_t$ is independent of the time parameterization of $C$, hence the name geometric phase. Following Simon [17], the integrand in (3) may be regarded as a connection 1-form $A^{(n)}$,

$$A^{(n)} = A^{(n)}_\mu d\xi^\mu = i\langle n, \xi | d | n, \xi \rangle = i\langle n, \xi | \frac{\partial}{\partial \xi^\mu} | n, \xi \rangle d\xi^\mu, \quad (4)$$

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dictating 'parallel transport' in a $U(1)$ bundle with $\xi$-space as a base and the phase of $|\psi_i\rangle$ in the fiber. The corresponding curvature 2-form $K^{(n)}$ is given by

$$K^{(n)} = dA^{(n)} = \frac{1}{2} K_{AB}^{(n)} \omega^A \wedge \omega^B, \quad \text{with} \quad K_{AB}^{(n)} = -2 \text{Im} (\partial_A \langle n, \xi \rangle (\partial_B |n, \xi\rangle))$$

(5)

(with $\partial_A \equiv \partial/\partial \xi^A$, etc) so that when the curve $C$ is closed, and the parameter space is topologically trivial, Stokes' theorem allows the expression of the geometric phase as a surface integral,

$$\gamma_C^{(n)} = \int_S K^{(n)}$$

(6)

where $S$ is any two-dimensional patch with $C$ as its boundary (see [1, 2, 16, 17]). It is worth remarking at this point that the phase $\gamma^{(n)}$ above is not necessarily the only contribution to the total geometric phase accumulated by the wavefunction. For certain types of Hamiltonian, the dynamic phase factor may also contain a geometric part, a fact that has been alluded to by Berry (see a remark at the beginning of section 6 of [3]) but seems otherwise to be little known.

When the initial state belongs to a degenerate multiplet, spanning a subspace $D$ of the total Hilbert space of the system, and the degeneracy persists for all values of the parameters along $C$, nothing prevents the state vector from mixing up with other states with the same energy. The connection 1-form $A^{(n)}$ mentioned above generalizes in this case to a matrix $A^{(D)}$ of 1-forms, with entries given by

$$A^{(D)}_{ef} = i (\epsilon, \xi | df, \xi)$$

(7)

where $\epsilon$ and $f'$ range over the degenerate eigenkets. The unitary matrix $U^{(D)}_C$ effecting the mixing is then given by

$$U^{(D)}_C = \mathcal{P} \exp \left( i \int_C A^{(D)} \right)$$

(8)

as shown by Wilczek and Zee [26]. When $A^{(D)}$ is diagonal at all points along $C$, the path-ordered exponential in (8) reduces to the standard exponential of the integral of $A^{(D)}$, and each degenerate eigenstate behaves exactly as in the non-degenerate case, acquiring the standard geometric phase of equation (3).

3. Quantum mechanics on embedded manifolds

Consider a quantum spinless particle constrained to move on a closed curve $r(s)$ embedded in three-dimensional Euclidean space—$s$ is an arbitrary parameter along the curve, not necessarily arclength. The confinement is implemented by a steep two-dimensional harmonic oscillator potential $U$ in the plane normal to the curve, of width $\eta$, and centered on the curve (see, e.g., [15]). Assuming a nowhere vanishing curvature $\kappa(s)$ for the curve, we define an adapted orthogonal frame given by the tangent, normal and binormal vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ at each point of the curve, where $\rho$ is the Euclidean norm of the tangent, $\rho = |\partial r/\partial s| = |\partial \ell/\partial s|$, $\ell$ being arclength, so that the triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is orthonormal. To extend this frame to a neighborhood of the curve, we write the position vector $\mathbf{x}$ of a point in $\mathbb{R}^3$, close enough to the curve, as

$$\mathbf{x}(s, a, b) = \mathbf{r}(s) + a \mathbf{n}(s) + b \mathbf{b}(s),$$

(9)

with $a$ and $b$ distances along $\mathbf{n}$ and $\mathbf{b}$, $(s, a$ and $b)$ are now coordinates adapted to $r(s)$, and the corresponding tangent vectors $\partial_r$, $\partial_a$ and $\partial_b$ provide a non-orthonormal, in general, adapted frame. Taking the width $\eta$ of the confining potential to be small, in the scale set by the curvature $\kappa$, $\eta \ll \kappa^{-1}$, guarantees that the adapted coordinate system is well defined in the region of
physical interest, i.e., where the wavefunction takes appreciable values. It is then convenient to introduce rescaled coordinates, \((a, b) \rightarrow (\eta a, \eta b)\), with \(\alpha\) and \(\beta\) of order 1, in the region of interest, and the corresponding frame \((\partial_\alpha, \partial_\beta, \partial_\psi)\). We now start from (9) and differentiate it, in turn, w.r.t. \(s, \alpha\) and \(\beta\), using \(\partial_\ell = \rho \partial_\ell\) (we remind the reader that \(\ell\) denotes arclength), as well as the Frenet–Serret formulas

\[
\partial_\ell t = \kappa n, \quad \partial_\ell n = -\kappa t + \tau b, \quad \partial_\ell b = -\tau n, \quad (10)
\]

and find

\[
\partial_\ell = \rho (1 - \eta \alpha \kappa) t - \eta \rho \tau \beta n + \eta \rho \alpha \tau b, \quad \partial_\alpha = \eta n, \quad \partial_\beta = \eta b, \quad (11)
\]

where \(\tau\) is the torsion of the curve, and \(t, n, b\) are considered differential operators. Note that we have expressed above the adapted frame \((\partial_\alpha, \partial_\beta, \partial_\psi)\), at \((s, \alpha\) and \(\beta)\), in terms of the Frenet triad \((t, n, b)\) at the corresponding (same \(s\)) point of the curve, using the Euclidean parallel transport. The Euclidean metric in the (rescaled) adapted coordinate system is given by

\[
G_{\alpha \beta} \equiv \partial_\alpha x \cdot \partial_\beta x, \quad (12)
\]

where partial derivatives are taken w.r.t. adapted (rescaled) coordinates \((s, \alpha, \beta)\) and the dot product is the Euclidean one on \(\mathbb{R}^3\)—explicitly it is found that

\[
G_{\alpha \beta} = \begin{pmatrix}
\rho^2 (1 - \eta \alpha \kappa)^2 + \eta^2 \rho^2 \tau^2 (\alpha^2 + \beta^2) & -\eta^2 \rho \beta \tau & \eta^2 \\
-\eta^2 \rho \beta \tau & \eta^2 & 0 \\
\eta^2 \rho \alpha \tau & 0 & \eta^2
\end{pmatrix}, \quad (13)
\]

with determinant \(G\) given by

\[
G = \eta^4 \rho^2 (1 - \eta \alpha \kappa)^2. \quad (14)
\]

The dynamics of the particle is governed by a 3D Hamiltonian \(H\), consisting of the standard kinetic part and the normal confining potential \(U\),

\[
H = -\frac{\hbar^2}{2m} \nabla^2 + U, \quad (15)
\]

the form of the latter being, in general, quite complicated when expressed in Cartesian coordinates. On the other hand, in terms of adapted coordinates, and setting \(\hbar\) as well as the particle’s mass equal to 1, we have

\[
H = -\frac{1}{2|G|^{1/2}} \partial_\alpha G^{\alpha \beta} |G|^{1/2} \partial_\beta + \frac{(\alpha^2 + \beta^2)}{2\eta^2}. \quad (16)
\]

Note that at the classical level, the motion of the particle along the curve is free, as \(U\) does not depend on \(s\). The corresponding normalization condition is \(\int ds \partial_\alpha G^{\alpha \beta} |G|^{1/2} \partial_\beta = 1\), where \(\Phi\) is the 3D wavefunction of the particle. It will prove convenient to work with a rescaled wavefunction \(\Psi \equiv e^{-h} \Phi\), with \(e^{-h} = (G/g)^{1/4}\) and \(g = \rho^2\) the metric of the curve, so that the normalization condition, in terms of \(\Psi\), becomes \(\int ds \partial_\alpha G^{\alpha \beta} g^{1/2} |\Psi|^2 = 1\), and \(\int ds \partial_\alpha G^{\alpha \beta} |\Psi|^2\) can be interpreted as the effective 1D probability density for the particle at the position \(s\) along the curve, using the conventional measure \(g^{1/2} ds\). Accordingly, the Hamiltonian undergoes the similarity transformation

\[
H \rightarrow \tilde{H} = e^{-h} He^h, \quad (17)
\]

which guarantees that if \(\Phi\) satisfies \(H\Phi = E\Phi\), then \(\Psi\) satisfies \(\tilde{H}\Psi = E\Psi\).
Our assumption of a small \( \eta \), compared to the curve’s radius of curvature, suggests a perturbative expansion of the ‘rescaled’ Hamiltonian \( \tilde{H} \). Thus, if the curvature \( \kappa(s) \) is of the order of \( \kappa_0 \), then the small dimensionless parameter of the expansion can be taken to be \( \kappa_0 \eta \). For a closed curve with a typical curvature \( \kappa_0 \approx R^{-1} \), the total length may be assumed of the order of \( R \), so that the energy associated with the motion of the particle along the wire is of the order of \( p^2 \sim R^{-2} \). On the other hand, the energy associated with the 2D harmonic oscillator in the normal plane is of the order of \( \hbar \omega \sim \eta^{-2} \), since the oscillator’s frequency \( \omega \) is equal to \( \eta^{-2} \). Thus, the dimensionless Hamiltonian \( \eta^2 \tilde{H} \), which expresses the total energy of the particle in units of the normal oscillator’s energy scale, may be expanded in a power series in \( \eta/R \).

The first term in such an expansion will be the (dimensionless) oscillator’s Hamiltonian, followed by corrections associated with the bending of the curve, and the particle’s tangential motion. Taking \( R = 1 \), so that (with a certain abuse of notation) \( \eta = \eta/R \) is now dimensionless, one finds

\[
\tilde{H} = \frac{1}{\eta^2} \tilde{H}_{-2} + \tilde{H}_0 + \eta \tilde{H}_1 + O(\eta^2),
\]

with

\[
\tilde{H}_0 = -\frac{1}{2} \frac{D^2}{\eta^2} - \frac{1}{8} \kappa^2
\]

\[
\tilde{H}_1 = -\kappa \alpha \frac{D^2}{\eta^2} - (r \kappa' + r\kappa) \alpha + \kappa \tau \beta D_r
\]

\[
-\frac{1}{2} ((r^2 + \kappa^2) + 3 r r' \kappa' + (r')^2 \kappa + r^2 \kappa'' + 2 r r'' \kappa) \alpha + (r \kappa' + 2 r\kappa \tau) \beta,
\]

where \( r \equiv \rho^{-1} \), \( D_r \equiv r \partial_\kappa - \tau L \) is a covariant derivative w.r.t. arclength, \( L = \alpha \partial_\alpha - \beta \partial_\beta \) is the generator of rotations in the normal plane and primes denote derivatives w.r.t. \( s \). The above expressions for \( \tilde{H}_{-2}, \tilde{H}_0 \) can be found in, e.g., [15]—the one for \( \tilde{H}_1 \) is new, as far as we know.

Writing the rescaled 3D wavefunction \( \Psi \) as

\[
\Psi(s, \alpha, \beta) = \chi(\alpha, \beta) \psi(s)
\]

and taking \( \chi = \chi_{\sigma n} \), with \( \chi_{\sigma n} \) simultaneous eigenkets of \( \tilde{H}_{-2} \) and \( L \),

\[
\tilde{H}_{-2} \chi_{\sigma n} = (n + 1) \chi_{\sigma n}, \quad L \chi_{\sigma n} = i \sigma \chi_{\sigma n},
\]

where \( n = 0, 1, 2, \ldots \) and \( \sigma = -n, -n + 2, \ldots, n - 2, n \), reduces the 3D eigenvalue relation \( \tilde{H} \Psi = \mathcal{E} \Psi \) to the 1D equation\(^6\)

\[
H_0 \psi_\sigma \equiv -\frac{1}{2} (\rho^{-1} \partial_\rho - i \sigma \tau)^2 \psi_\sigma - \frac{1}{8} \kappa^2 \psi_\sigma
\]

\[
= -\frac{1}{2 \rho^2} \psi_\sigma'' + \left( i \sigma \rho^{-1} \tau + \frac{\rho'}{2 \rho} \right) \psi_\sigma' + \frac{1}{2} \left( i \sigma \rho^{-1} \tau + \sigma^2 \tau^2 - \frac{1}{4} \kappa^2 \right) \psi_\sigma
\]

\[
= \mathcal{E}_\sigma \psi_\sigma
\]

for the tangential wavefunction to order \( O(\eta^0) \)—higher order corrections, due to \( \tilde{H}_1 \) etc, can be treated perturbatively. Note the appearance of the induced potential term \( -\kappa^2/8 \) on the rhs of (24), due to which particles will tend to accumulate in regions of higher curvature. For a classical particle, confined exactly on the curve, such a term would be absent, as the only geometrical characteristics of the curve it would be able to detect would be intrinsic

\(^5\) The spectrum of \( L \) can be consulted in, e.g., [13], chapter 12, section 14.

\(^6\) Note that the 1D Hamiltonian \( H_0 \) is obtained from the 3D one \( H_0 \) of equation (20) by substituting for \( L \) its eigenvalue \( i \sigma \).
ones (similar statements hold true for confinement on surfaces, etc). But the purely quantum phenomenon of the particle’s wavefunction penetrating a small, but finite, distance into the plane normal to the curve allows for the detection of extrinsic characteristics, such as, e.g., the curvature $\kappa$. As a result, an infinite straight wire, for example, which is bent just once along its length, admits bound states in the vicinity of the bend.

The existence of closed space curves of arbitrarily small total torsion $\phi(s) \equiv \int_0^s \tau(s') \, ds'$ [25] shows that the phase redefinition of the wavefunction by $e^{i\phi(s)}$, which would gauge away the torsion term in (24), cannot, in general, be effected globally. Putting everything together, we see that the $\tilde{H}$ eigenstates can be written, up to $O(\eta^0)$, as

$$\Psi_{\sigma nm} = \chi_{\sigma n} \psi_{\sigma m}$$

with energy

$$E_{\sigma nm} = \frac{(n+1)}{\eta^2} + E_{\sigma m},$$

where, due to the periodic boundary conditions in $s$, the index $m$ labeling the solutions $\psi_{\sigma m}$ of (24) is discrete. The structure of equation (24) implies that $\psi_{-\sigma m} = \psi_{\sigma m}^*$ and $E_{-\sigma m} = E_{\sigma m}$. This, together with the fact that $\chi_{-\sigma n} = \chi_{\sigma n}^*$, leads to

$$\Psi_{-\sigma nm} = \Psi_{\sigma nm}^* \quad \text{and} \quad E_{-\sigma nm} = E_{\sigma nm}. \quad (27)$$

As a conclusion, unless $\sigma = 0$, there is at least a twofold degeneracy, with extra degeneracy possible in the tangential wavefunction $\psi_{\sigma m}$. Since (27) holds for any curvature and torsion of the curve, it is satisfied throughout the deformations we contemplate, so that the associated Berry curvature satisfies

$$K_{-\sigma nm} = -K_{\sigma nm}, \quad (28)$$

as is easily inferred from (5). These conclusions hold only up to $O(\eta^0)$—we see no reason why they should persist when higher orders are considered.

4. The geometric phase in an inertial frame

4.1. The parameter space of a particle constrained on a curve

A look at (24) shows that the shape of the curve on which the quantum particle is constrained, codified in the two functions $\kappa(s)$ and $\tau(s)$, may be considered as an infinite family of parameters, which the effective 1D Hamiltonian governing the tangential motion depends on. Furthermore, $\rho(s)$ also plays the role of parameter for $H_0$, since the obvious reparametrization invariance of $H_0$ involves transforming $\partial_s$, as well, while in (24) only $\rho(s)$ changes when the curve is being deformed—of course, physical predictions do not depend on $\rho$. One may parametrize this infinite-dimensional parameter space, for example, by the Fourier modes $\kappa_m$, $\tau_n$, and $\rho_k$ of $\kappa(s)$, $\tau(s)$ and $\rho(s)$, respectively. However, not all of these parameters are independent: for example the reality of $\kappa(s)$ implies that $\kappa_{-m} = \kappa_m^*$, and similarly for $\tau_n$, $\rho_k$, while the closure condition for the curve further restricts these parameters in a nontrivial way. It is for this reason that we opt to employ the Fourier modes of the components of the position vector instead. If one considers now cyclic deformations of the curve, the corresponding point in the above infinite-dimensional parameter space traces a closed loop (the latter should not be confused with the closed curve the particle lives on). For each configuration of the curve, the 1D Schrödinger equation in (24) may be solved for $\psi_{\sigma m}(s)$, giving rise to the corresponding 3D wavefunction $\Psi_{\sigma m}(s)$ of (25), so that the physical wavefunction $\Phi_{\rho\sigma m} = (G/g)^{-1/4}\Psi_{\rho\sigma m}$ can finally be determined, as a function of the deformation parameters—then equation (5) will reveal whether geometric phases are present in the system under study.
4.2. Infinitesimal deformations and their induced vector field

We now study deformations of a curve, described by velocity vector fields defined along the curve. In this section, we focus on a single vector field and its associated deformation parameter. We derive expressions for the induced 3D velocity field of the adapted coordinates and its expansion in powers of \( \eta \). In the subsequent sections, starting with section 4.5, we will consider a pair of fields, so that we can describe closed curves in parameter space, enclosing nonzero area—a necessary condition for the appearance of geometric phases.

An arbitrary infinitesimal deformation of a curve \( \mathbf{r}(s) \) can be described by a vector field \( \mathbf{v}(s) \) defined along the curve, specifying the initial velocity of each point of the curve under the deformation. However, for our perturbative treatment of the Schrödinger equation to be valid, the displacement of the curve due to its deformation must be much smaller than the width of the wavefunction in the normal plane, which is of the order of \( \eta \). Thus, we find it convenient to include explicitly a factor of \( \eta \) in the deformation,

\[
\mathbf{r}(s, \xi(t)) = \mathbf{r}(s) + \eta \xi(t) \mathbf{v}(s),
\]

where the smallness of the deformation parameter is attributed to a number \( \lambda \), \( \xi(t) = \lambda \xi(t) \), with \( \lambda \ll 1 \) and \( \xi(t) \) of order 1, and dots denote time derivatives. Written in this form, our results will hold for every \( \xi \ll 1 \), regardless of the value of \( \eta \). An important class of deformations is given by the locally arclength preserving (LAP) ones, which, in terms of Frenet frame components, \( \mathbf{v} = \mathbf{v}^t + \mathbf{v}^n \mathbf{n} + \mathbf{v}^b \mathbf{b} \), satisfy

\[
\mathbf{v}_t = \mathbf{v}_t \cdot \mathbf{t} = \mathbf{v}_e - k \mathbf{n} = 0, \tag{30}
\]
a relation that, when valid, may be used to further simplify our results (in the above expression \( \mathbf{v}_t \) denotes the \( t \)-component of the covariant derivative of \( \mathbf{v} \) w.r.t. arclength \( \ell \), while \( \mathbf{v}_e \) is the partial derivative, w.r.t. \( \ell \), of the component \( \mathbf{v}^e \) —we use a similar notation for higher order derivatives, e.g., \( \mathbf{v}_{ee} \) for the second-covariant derivative, etc.). In what follows, our results hold true for arbitrary deformations, except when explicitly stated that the LAP condition has been used.

An infinitesimal deformation, of the form \( \mathbf{r} \to \mathbf{r} + \eta \xi \mathbf{v} \), with \( \xi \ll 1 \), brings along changes \( \eta \xi \rho \iota, \eta \xi \kappa_\xi, \eta \xi \tau_\xi \), in \( \rho, \kappa, \tau \), respectively,

\[
\rho_\xi = \rho \mathbf{v}_\xi = \rho \mathbf{v}_e \tag{31}
\]

\[
\kappa_\xi = \mathbf{v}_\ell \cdot \mathbf{n} - 2 \kappa \mathbf{v}_\ell \tag{32}
\]

\[
\tau_\xi = \kappa^{-1} \mathbf{v}_\ell (\mathbf{v}_\ell \cdot \mathbf{n}) - k \mathbf{v}_\ell \cdot \mathbf{n} - \kappa^{-2} \kappa \mathbf{v}_\ell \cdot \mathbf{n} + \kappa \mathbf{v}_\ell, \tag{33}
\]

which in turn result in the perturbation Hamiltonian \( \eta \xi H^\xi \), with

\[
H^\xi = \frac{\partial_\xi}{\rho^3} \frac{\partial_\xi}{\partial_\xi} + \left( \frac{i \sigma \tau_\xi}{\rho} - \frac{i \sigma \rho_\xi}{2 \rho^3} + \frac{\rho_\xi}{2 \rho^3} - \frac{3 \rho_\xi}{2 \rho^3} \right) \partial_\xi + \frac{i \sigma \tau_\xi}{2 \rho} - \frac{i \sigma \rho_\xi}{2 \rho^2} - \frac{\rho_\xi}{2 \rho^2} + \frac{\sigma^2 \tau_\xi}{4} - \frac{1}{4} \kappa \kappa_\xi, \tag{34}
\]

to be added to the unperturbed Hamiltonian \( H_0 \) of equation (24). The perturbed wavefunctions, as mentioned above, are to be inserted in (5) for the calculation of Berry’s curvature. However, these wavefunctions are expressed in the adapted coordinates, which depend implicitly on the deformation parameters, since the curve being deformed drags with it the adapted frame. Our task then, in principle, would be to express the adapted coordinates in terms of the Cartesian ones, differentiate them with respect to the parameters, treating the Cartesian coordinates as constants, and reexpress the result in terms of the adapted coordinates. Then the derivatives with respect to the parameters in, e.g., equation (5), should contain contributions both from
the explicit dependence of $\Phi(s, \alpha, \beta; \xi)$ on the parameters and from the implicit one through the adapted coordinates,

$$\frac{d\Phi}{d\xi} = \frac{\partial \Phi}{\partial \xi} + \frac{\partial s}{\partial \xi} \frac{\partial \Phi}{\partial s} + \frac{\partial \alpha}{\partial \xi} \frac{\partial \Phi}{\partial \alpha} + \frac{\partial \beta}{\partial \xi} \frac{\partial \Phi}{\partial \beta},$$

i.e., we may write the total $\xi$-derivative as

$$\frac{d}{d\xi} = \frac{d}{d\xi} + \xi \eta \mathbf{U},$$

with

$$\eta \mathbf{U} = \frac{\partial s}{\partial \xi} \mathbf{e}_s + \frac{\partial \alpha}{\partial \xi} \mathbf{e}_\alpha + \frac{\partial \beta}{\partial \xi} \mathbf{e}_\beta,$$

so that $\eta(0) \mathbf{U}$ is the initial ($t = 0$) velocity field, under the deformation, of points with fixed Cartesian coordinates, as seen in the adapted coordinate frame. The 3D vector field $\mathbf{U}$ may be easily related to the 1D deformation velocity field along the curve, $\mathbf{v}(s)$. Indeed, describing the deformation of the curve by $\mathbf{r}(s, \xi)$, with $\mathbf{v}(s) = \partial_s \mathbf{r}(s, \xi) |_{\xi=0} = \partial_s \mathbf{r}(s, 0)$, we obtain

$$\mathbf{x}(s, \alpha, \beta, \xi) = \mathbf{r}(s, \xi) + \eta \alpha \mathbf{n}(s, \xi) + \eta \beta \mathbf{b}(s, \xi).$$

Keeping the adapted coordinates constant, we now define

$$\eta \mathbf{V}(s, \alpha, \beta) = \partial_s \mathbf{x}(s, \alpha, \beta, 0) = \eta \mathbf{v}(s) + \eta \alpha \partial_s \mathbf{n}(s, 0) + \eta \beta \partial_s \mathbf{b}(s, 0),$$

so that $\eta(0) \mathbf{V}$ is the initial velocity field, under the deformation, of points with fixed adapted coordinates, observed in the ambient $\mathbb{R}^3$, as they are dragged along by the moving curve, and is therefore the opposite of $\eta(0) \mathbf{U}$, i.e., $\mathbf{V} = -\mathbf{U}$.

A digression on notation is in order at this point. In the above expressions, we have expanded vector fields in the $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ frame. The latter is initially defined along the curve, but can be extended to a 3D neighborhood of it by Euclidean parallel transport to all points of the corresponding normal plane. As we have already mentioned, our calculations will involve expansions in powers of $\eta$, so it would be convenient to exhibit all $\eta$-dependence explicitly. While this is straightforward in the case of functions, it warrants a moment of thought when differential operators, like the above vector fields, are concerned. This is because the order in $\eta$ of a differential operator depends also on the functions it is applied to. Thus, one might consider $\mathbf{n} = \partial / \partial \alpha$ as zeroth order in $\eta$, but when applied to the normal wavefunction $\chi(\alpha)^{(n)}$, of (23), the result is of the order of $\eta^{-1}$. With this in mind, we will expand, in what follows, vector fields operating on wavefunctions in the rescaled adapted frame $(\partial_{\xi}, \partial_{\alpha}, \partial_{\beta})$, all basis elements of which are of the order of $\eta^0$, in the above sense—the relation of this frame with the 3D extension of the Frenet triad mentioned above is given by (11). On the other hand, we assume velocity fields of zeroth order in $\eta$, so we write $\mathbf{v}^g = \eta^{-1} \mathbf{v}^a$, $\mathbf{v}^b = \eta^{-1} \mathbf{v}^b$, with $\mathbf{v}^a$, $\mathbf{v}^b \sim O(\eta^0)$.

To relate $\mathbf{V}(s, \alpha, \beta)$ to $\mathbf{v}(s)$, we calculate

$$\partial_s \mathbf{t} = \eta(\mathbf{v}_s \mathbf{n} + \mathbf{v}_b), \quad \partial_s \mathbf{n} = \eta(-\mathbf{v}_s \mathbf{t} + \mathbf{v}^b), \quad \partial_s \mathbf{b} = \eta(-\mathbf{v}_s \mathbf{t} - \mathbf{v}^b),$$

where $\mathbf{v}^0 = \kappa^{-1} \mathbf{v}^0$. Using these expressions, equation (38) gives

$$\mathbf{V} = -\mathbf{U} = (\mathbf{v}^g - \eta \alpha \mathbf{v}^a - \eta \beta \mathbf{v}^b) \mathbf{t} + (\mathbf{v}^b - \eta \beta \mathbf{v}^0) \mathbf{n} + (\mathbf{v}^b + \eta \alpha \mathbf{v}^0) \mathbf{b}$$

$$= \mu \rho^{-1} \partial_s + (\eta^{-1} \mathbf{v}^0 - (\mathbf{v}^0 + \mu \mathbf{t}) \mathbf{v}^0) \partial_s + (\eta^{-1} \mathbf{v}^b + (\mathbf{v}^0 - \mu \mathbf{t}) \mathbf{v}^b) \partial_s,$$

where $\mu = (\mathbf{v}^0 - \eta \alpha \mathbf{v}^a - \eta \beta \mathbf{v}^b)/(1 - \eta \mathbf{a})$—note that this is an exact result for the general $(\rho \neq 1$, non-LAP) case. The infinitesimal displacement generated by $\mathbf{V}$ is a rigid motion of the planes normal to the curve, as is clearly reflected in the above result: referring to (40), the $(\mathbf{v}^g, \mathbf{v}^a, \mathbf{v}^b)$ part is just the velocity of the curve itself. The $\eta \mathbf{v}^0(0, -\beta, \alpha)$ part is responsible for rotating the planes around $\mathbf{t}$, while the $-\eta(\alpha \mathbf{v}^a + \beta \mathbf{v}^b)$ correction to the $\mathbf{t}$-component comes from the rotation of the normal planes, around an axis lying in them, as they follow the
curve, the latter rotating due to the normal component of $v_i$. In our perturbative calculations, the expansion of $V$ in powers of $\eta$ will be useful,

$$V = \frac{1}{\eta} V_{-1} + V_0 + \eta V_1 + \mathcal{O}(\eta^2),$$  

(42)

with

$$V_{-1} = (0, v^\alpha, v^\beta)$$  

(43)

$$V_0 = (v^\alpha, -(v^\beta - \tau v^\gamma)\beta, (v^\alpha - \tau v^\gamma)\alpha)$$  

(44)

$$V_1 = (\tau((v^\alpha - \kappa v^\gamma)\alpha + v^\beta \beta), -\tau(v^\alpha - \kappa v^\gamma)\alpha \beta - \tau v^\beta \beta^2, \tau(v^\alpha - \kappa v^\gamma)\alpha^2 + \tau v^\beta \alpha \beta).$$  

(45)

### 4.3. A note on hermiticity

In the calculations that follow, we consider inner products in the Hilbert space of rescaled particle wavefunctions $\Psi$, either with unit measure (original, undeformed curve, parameterized by arclength) or with measure $\rho(s, t)$ ($\xi$-deformed curve, with $s$-parameterization carried over from the undeformed one). The corresponding Hermitian conjugation for $\partial_s$ is

$$\partial_s^\dagger = -\partial_s - \rho^{-1}\rho,$$  

(46)

respectively, the rest of the basic operators having the standard Hermitean conjugates in both cases. Norm conservation, on the other hand, implies, in the general case of a time-dependent measure $\rho$,

$$H^\dagger - H = i\rho^{-1}\rho,$$  

(47)

for the total Hamiltonian $H$ of the system. Thus, for a nontrivial ($s$-dependent) but constant in time measure $\rho(s)$, $H$ is Hermitean w.r.t. $\cdot^\dagger$. When deformations are considered, with $\rho(s, t) = 1 + \xi(t)\chi^\xi(s) + \mathcal{O}(\xi^2)$, where $\chi^\xi = v^\xi_i$ (see (31)), one obtains, to first order in $\xi$,

$$\partial_s^\dagger = -\partial_s - \xi \chi^\xi,$$  

(48)

$$H^\dagger - H = i\xi \chi^\xi.$$  

(49)

In what follows the hermiticity properties of $H^\xi$, defined in (34), and of $\mathcal{H}^\xi$, defined as

$$\eta \mathcal{H}^\xi \equiv i(\eta V^\xi - d_\xi h) = i\eta V^\xi - d_\xi h + \eta V^\xi(h)$$  

will be needed ($V^\xi$ is given above, in (40), (41), while $h$ is the logarithm of the wavefunction rescaling factor—see the paragraph before (17)). A short calculation shows that $d_\xi h = -\eta \kappa v^\xi / 2 + \mathcal{O}(\eta^2)$, so that

$$\mathcal{H}^\xi = i(V^\xi + \frac{1}{2}\kappa v^\xi) + \mathcal{O}(\eta).$$  

(50)

Assume that $H = H_0 + \xi H^\xi$ is Hermitean w.r.t. the measure $\rho = 1 + \xi \chi^\xi + \mathcal{O}(\xi^2)$, i.e., $H^\dagger = H = 0$, while $H_0$ is Hermitean w.r.t. $\rho = 1, H^\dagger_0 - H_0 = 0$. It is easily shown that, under these circumstances, the perturbation $H^\xi$ satisfies

$$H^\dagger - H^\xi = [\chi^\xi, H_0],$$  

(51)

which, upon substitution of $H_0$ from (24), with $\rho = 1$, gives

$$H^\dagger - H^\xi = \chi^\xi \partial_s + \frac{1}{2} \kappa v^\xi - i\sigma \tau \chi^\xi,$$  

(52)

a relation that $H^\xi$ of (34) indeed satisfies. On the other hand, a straightforward calculation, starting with definition (49), shows that

$$\mathcal{H}^\xi - \mathcal{H}^\xi = i\eta \chi^\xi + \mathcal{O}(\eta^2).$$  

(53)

Note that, to leading order in $\eta$, $\mathcal{H}^\xi$ is Hermitean, iff the corresponding deformation is LAP. From (51), (53), it may be inferred that

$$H^\xi_{nk} = H^\xi_{nk} - E_{nk} \chi^\xi_{nk}, \quad \mathcal{H}^\xi_{nk} = \mathcal{H}^\xi_{nk} + i\chi^\xi_{nk},$$  

(54)

where $E_{nk} \equiv \mathcal{E}_n - \mathcal{E}_k$, $n, k$, being here composite indices.
4.4. Remarks on Berry’s connection

As remarked earlier, the spectrum of $\tilde{H}$ is at least twofold degenerate, corresponding to opposite eigenvalues $\pm i\sigma$ of $L$. We will now show that when no additional degeneracy is present, the Wilczek–Zee connection matrix of equation (7) is diagonal, to lowest nonvanishing order in the deformation parameter $\lambda$. Indeed, in this case, the two degenerate states differ only in the sign of their $L$ eigenvalue, having in common both the occupation number $n$ of the normal harmonic oscillator and the index $m$ of the tangential wavefunction, so that (7) gives

$$\left(\Lambda_\xi\right)_{\sigma\zeta} = i \int d^3x \Phi_{\zeta m}(x; \xi, \eta) \frac{d}{d\xi} \Phi_{\sigma m}(x; \xi, \eta), \quad (55)$$

where $\zeta = \pm \sigma$ and we have suppressed the indices $n$ and $m$ in $A$. Switching to rescaled adapted coordinates $(s, \alpha \equiv y^1, \beta \equiv y^2)$, using $d/d\xi = \partial_\xi - \eta V$, and expanding in powers of $\eta$, results in

$$\left(\Lambda_\xi\right)_{\sigma\zeta} = i \int ds^2 y \left(1 - \frac{1}{2} i \eta \sigma \kappa \right) \psi_m^* (s; \xi) \chi_n^* (y)$$

$$\left(\partial_\xi - \eta V \partial_\alpha - \eta \partial_\beta - \eta \partial_\eta \right) \left(1 + \frac{1}{2} i \eta \sigma \kappa \right) \psi_m (s; \xi) \chi_n (y)$$

$$= -i \int ds \psi_m^* \psi_{\sigma m} \int d^2 y \chi_n^* \partial_\alpha \chi_n - i \int ds \psi_m^* \psi_{\sigma m} \int d^2 y \chi_n^* \partial_\beta \chi_n$$

$$+ i \int ds \psi_m^* \partial_\eta \psi_{\sigma m} \int d^2 y \chi_n^* \chi_n - i n \int ds \psi_m^* \psi_{\sigma m} \int d^2 y \chi_n^* L \chi_n$$

$$- i n \int ds \psi_m^* \chi_n \psi_{\sigma m} \int d^2 y \chi_n^* \chi_n - \frac{i n}{2} \int ds \psi_m^* \chi_n \psi_{\sigma m} \int d^2 y \chi_n^* \chi_n \quad (56)$$

It is easily shown that the operators $\partial_\alpha, \partial_\beta$, appearing in the first two terms, only connect harmonic oscillator eigenstates the $L$ eigenvalues of which differ by $\pm 1$, while, in our case, they differ by $0, \pm 2$, etc, so these terms vanish. The third term is due to the explicit $\xi$-dependence of the tangential wavefunctions and would provide the entire Wilczek–Zee connection for an observer oblivious to the $\xi$-dependence of the adapted coordinates, the latter giving rise to the last three terms. Taking into account the vanishing of the first two terms, and the fact that the $\chi$ are $L$-eigenfunctions and orthonormal, we conclude that the matrix $\Lambda_\xi$ is diagonal in the degenerate subspace. In the next section, when we compute Berry’s curvature, we will make use of this fact to simplify our computations.

The above expression may be simplified if the LAP condition is used,

$$\left(\Lambda_\xi\right)_{\sigma\zeta} = \delta_{\sigma\zeta} \int ds \psi_m^* \left(i \partial_\xi + \eta \left(\sigma V^0 - i V^j \partial_j - \frac{i}{2} \kappa V^0\right)\right) \psi_{\sigma m} \quad (57)$$

where, in the second line, we separated the real and the imaginary part, and used the LAP condition to show that the latter vanishes, being the integral of a total derivative (note that $\partial_\xi$ acts to the left on $\psi_m^* \psi_{\sigma m}$). Similar expressions can be obtained for Berry’s curvature, but the results are rather complicated and do not seem particularly illuminating.

4.5. Berry’s curvature in Cartesian coordinates

We are now ready to use (5), with total rather than partial derivatives, as explained above, to calculate Berry’s curvature. Written out explicitly, in terms of the original, unscaled wavefunction $\Phi_n$, the above expression becomes

$$K_{\xi\zeta}^{(n)} = i \int ds^2 \sqrt{G} \left[\left(\partial_\xi - \eta V^j\right) \Phi_n^* \left(\partial_\xi - \eta V^j\right) \Phi_n\right] - \text{c.c.} \quad (58)$$
Substituting $\Phi_0 = e^{i\xi}\Psi_0$, and commuting the exponential factors through the differential operators to join the integration measure, results in

$$K^{(n)}_{\xi} = i \int ds d^2\sqrt{\gamma}(\partial_\xi + i n \mathcal{H}^\xi)\Psi_n)(\partial_\xi + i n \mathcal{H}^\xi)\Psi_n] - c.c.,$$

with $\mathcal{H}^\xi$ defined in (49). The first-order corrected rescaled wavefunction $\Psi_n$, in the presence of the perturbation $n \xi \mathcal{H}^\xi$ of (34), is given by $\Psi_n(s, \alpha, \beta; \xi) = (s, \alpha, \beta|n, \xi)$, with

$$|n, \xi\rangle = |n\rangle + n \xi \left(-\frac{1}{2} \chi_{nn}|n\rangle + \sum_{k \neq n} E^{-1}_{nk} H_k^\xi|k\rangle\right),$$

therefore

$$\partial_\xi |n, \xi\rangle = -\frac{n}{2} \chi_{nn}|n\rangle + n \sum_{k \neq n} E^{-1}_{nk} H_k^\xi|k\rangle,$$

where $|n\rangle$ is assumed non-degenerate (the index $n$ is used here as a generic label for the particle’s state). Note the component of the first-order correction along $|n\rangle$, present only for non-LAP deformations, amounting to a change in normalization in order to compensate for the expansion of $s$-space. An analogous correction, due to $\mathcal{H}_1$, should in principle be included as well, but a glance at (21) shows that all terms of $\mathcal{H}_1$ are proportional to either $\alpha$ or $\beta$, and these operators only connect the zeroth-order state with different occupation number states in the normal plane, and, hence, the first-order correction to the wavefunction is suppressed by the energy denominator by an additional power of $n^2$, and may be neglected at low-order calculations.

For a pair of deformations, with the corresponding parameters $\xi$ and $\zeta$, the curvature is now readily computed from (59),

$$\eta^{-2} K^{(n)}_{\xi\zeta} = i \sum_{k \neq n} E^{-1}_{nk}(H^\xi_{nk}H_k^{\zeta} - H_{nk}^\xi H_k^{\zeta})$$

$$+ \sum_{k \neq n} E^{-1}_{nk}(H^\xi_{nk}H_k^{\zeta} + H_{nk}^\xi H_k^{\zeta} - H^\xi_{nk}H_k^{\zeta} - H_{nk}^\xi H_k^{\zeta})$$

$$+ i(H_{nk}^\xi H_k^{\zeta} - H_{nk}^\xi H_k^{\zeta})_{nn} + \frac{1}{2} \chi_{nn}(H_{nn}^\xi H_{nn}^{\zeta} + H_{nn}^{\xi} H_{nn}^{\zeta}) - \frac{1}{2} \chi_{nn}(H_{nn}^\xi H_{nn}^{\zeta} + H_{nn}^{\xi} H_{nn}^{\zeta}).$$

A few remarks are due at this point. We begin with the clarification that our intention is to compute $K_{\xi\zeta}$ to the lowest non-vanishing order only, in $\eta$, as this suffices to show that it is nonzero, in general. The number computed by (61), on the other hand, is the value of the curvature at the origin of the $\xi-\zeta$ plane, including contributions of various orders in $\eta$; thus, a power expansion of the rhs of (61) is called for. Another point worth clarifying is that the sums over $k$ in (61) are, in principle, over a complete set of states, except for the state $|n\rangle$. However, for states with different total occupation number of the 2D normal oscillator, the energy prefactor in both sums suppresses the corresponding contribution by a factor of $n^4$, in the first sum, and $n^2$, in the second. Thus, the sums may be safely restricted to the tangential 1D modes, and to the spectrum of $\sigma$, skipping $|n\rangle$. But, as shown below, all matrix elements involved are diagonal in $\sigma$; hence, the sums may be further restricted to all tangential modes with the same angular momentum in the normal plane as the unperturbed state, except itself. Note also that, by its very definition, the calculation of $K_{\xi\zeta}$ at the origin of the $\xi-\zeta$ plane involves a limiting process $\lambda \to 0$, so that only leading order, in $\lambda$, terms need to be taken into account—in particular, $\sqrt{\lambda} = \rho = 1$ may be used in (59), assuming that the undeformed curve is parameterized by arclength.

We may further simplify (61), bringing it into a form that will facilitate comparison with the result of section 5, by making use of (54). Various cancellations take place, the nontriviality
of which is best appreciated by actually going through the algebra in detail. Use has to be made of the completeness relation $\chi^\xi_n \mathcal{H}^\xi_{nn} + \sum_{k \neq n} \chi^\xi_n \mathcal{H}^\xi_{kn} = (\chi^\xi \mathcal{H}^\xi)_{nn}$, etc—when the dust settles, our final formula for the curvature emerges,

$$
\eta^{-2} \mathcal{K}^{(\eta)}_{\xi \xi} = i \sum_{k \neq n} E^{-1}_{nk} \left( \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{kn} - \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{lk} \right) + \sum_{k \neq n} E^{-1}_{nk} \left( \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{kn} + \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{kn} - \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{kn} - \mathcal{H}^\xi_{nk} \mathcal{H}^\xi_{kn} \right) + i \left[ \mathcal{H}^\xi_{\xi}, \mathcal{H}^\xi \right]_{\text{int}}.
$$

\[62\]

5. The geometric phase in a noninertial frame

In section 3, we determined the form of the particle’s Hamiltonian in the (inertial) Cartesian coordinates $(t, x^1, x^2, x^3)$, and then expressed it in the (noninertial, in general) adapted coordinates $(t', s, \alpha, \beta) \equiv (t', y^1, y^2, y^3)$, equation (16), aiming at simplicity of form. The operator thus obtained is not the Hamiltonian $\mathcal{H}$ in the noninertial frame, the latter being defined as the generator of time evolution for the corresponding wavefunction. This new operator, $\tilde{\mathcal{H}}$, is obtained by transforming the time-dependent Schrödinger equation $\psi = -i \mathcal{H} \psi$, with $\mathcal{H}$ as in (16), to the adapted coordinates.

5.1. The form of the Hamiltonian in adapted coordinates

Under the change of coordinates $(t, x^1, x^2, x^3) \rightarrow (t', y^1, y^2, y^3)$, with $t' = t$ and $y' = y'(x; \xi (t))$, the time derivative in the time-dependent Schrödinger equation transforms as

$$
\partial_t = \partial_{t'} + \dot{\xi} (\partial_{y^1} / \partial \xi) \partial_{y^1} = \partial_{t'} - \eta \dot{\xi} \mathbf{V},
$$

(63)

where we have assumed that the time dependence of the coordinate transformation is entirely through $\xi (t)$ and $\mathbf{V}$ is given in terms of the deformation velocity field of the curve by (40) and (41). The $\dot{\xi}$-dependent term on the rhs of (63) is to be taken to the rhs of the time-dependent Schrödinger equation, thus giving for $\tilde{\mathcal{H}}$,

$$
\tilde{\mathcal{H}} = \mathcal{H} + i \eta \dot{\xi} \mathbf{V}.
$$

(64)

Next, the rescaling of the wavefunction is effected, with the corresponding similarity transformation for the Hamiltonian. However, the time dependence of the rescaling factor $e^{\dot{\xi}}$ means that the similarity transformation of the time derivative is nontrivial, $\partial_t \rightarrow e^{-\dot{\xi}} \partial_t e^{\dot{\xi}} = \partial_{t'} + \dot{\xi} \partial_{t'}$, resulting in a further $\dot{\xi}$-dependent contribution,

$$
\tilde{\mathcal{H}} = \tilde{\mathcal{H}} + i \dot{\xi} (\eta \tilde{\mathbf{V}} - \partial_{t'} d_\xi) = \tilde{\mathcal{H}} + i \dot{\xi} (\eta \tilde{\mathbf{V}} - d_\xi) \equiv \tilde{\mathcal{H}} + i \eta \dot{\xi} \tilde{\mathbf{H}}^\xi,
$$

(65)

where in the second equation, use was made of the fact that $\tilde{\mathbf{V}} = \mathbf{V} + \mathbf{V} \partial_{t'}$, and $\tilde{\mathcal{H}}, d_\xi$ are as in (18) and (36), respectively (with $\mathbf{U} \rightarrow -\mathbf{V}$).

Assuming, as before, an infinitesimal deformation of some initial curve, involving two parameters $\xi$ and $\zeta$, we arrive at

$$
\tilde{\mathcal{H}} = \tilde{\mathcal{H}} + i \eta \dot{\xi} \tilde{\mathcal{H}}^\xi + i \dot{\xi} (\eta \omega - d_\zeta) \equiv \tilde{\mathcal{H}} + i \eta \dot{\xi} \tilde{\mathcal{H}}^\xi + i \dot{\xi} (\eta \omega - d_\zeta),
$$

(66)

where $\omega$ is the 3D vector field corresponding to the $\zeta$-deformation. The novel feature in $\tilde{\mathcal{H}}$ is its dependence on $\dot{\xi}, \dot{\xi}$, the corresponding terms codifying the effect of inertial forces felt in the noninertial adapted frame. As a result, the standard formula for the calculation of the geometric phase, equation (5), fails in this case, as part of the geometrical phase resides in
the dynamical phase factor. Accordingly, we resort to time-dependent perturbation theory for the calculation—the fact that the geometric phase accumulated during a cyclic deformation scales like an area enclosed in parameter space, means that we have to keep up to quadratic terms in the deformation parameter. In particular, this ensures that quadratic corrections to the Hamiltonian are taken into account in the calculation of the quadratic part of its instantaneous eigenvalue and, hence, of the dynamical phase factor. In the rest of this section, we develop the general formulas needed for our purposes.

5.2. Time-dependent perturbation theory

Our starting point is the time-dependent Schrödinger equation \( \dot{\psi} = \partial_t \psi \), etc
\[
\psi(t) = -i[H + \lambda V(t) + \lambda^2 \tilde{Q}(t)]\psi(t), \tag{67}
\]
where \( H \) does not depend on time, \( \lambda \) is small and higher than quadratic terms are neglected. Factoring the time evolution operator as \( \psi(t) = e^{-iHt}U(t)\psi(0) \), we obtain from (67) that
\[
U(t) = -i(\lambda \tilde{V}(t) + \lambda^2 \tilde{Q}(t))U(t), \tag{68}
\]
with \( \tilde{V} \equiv e^{iHt}V(t)e^{-iHt} \), and similarly for \( \tilde{Q} \). Integrating, with the initial value \( U(0) = 1 \), and iterating the result, gives
\[
U(t) = 1 - i\lambda \int_0^t dt_1 \tilde{V}(t_1) - \lambda^2 \left( \int_0^t dt_1 \int_0^t dt_2 \tilde{V}(t_1)\tilde{V}(t_2) + i \int_0^t dt_1 \tilde{Q}(t_1) \right) + O(\lambda^3). \tag{69}
\]
Consider a non-degenerate eigenket \( |n\rangle \) of \( H, H|n\rangle = E_n|n\rangle \), and arrange for the perturbation to vanish at the start and finish of its period, \( V(0) = V(T) = 0 \). If, at \( t = 0 \), the system is put in the state \( |\psi(0)\rangle = |n\rangle \), then at the end of one cycle it will have evolved to
\[
|\psi(T)\rangle = e^{-iHt}U(T)|n\rangle. \tag{70}
\]
Assuming adiabaticity, \( |\psi(T)\rangle \) will be proportional, by a phase factor, to \( |n\rangle \), \( |\psi(T)\rangle = e^{i\gamma(n)}e^{-iE_nT}|n\rangle \), where we have factored out explicitly the dynamical phase factor due to the unperturbed energy, and the residual phase \( \gamma(n) \) is given by
\[
e^{i\gamma(n)} = \langle n|U(T)|n\rangle. \tag{71}
\]
Use of (69), and expansion in a complete set of states, gives
\[
e^{i\gamma(n)} \approx 1 - i\lambda \int_0^T dt_1 V_{mn}(t_1) - \lambda^2 \left( \sum_k \int_0^T dt_1 \int_0^T dt_2 e^{iE_k(t_1-t_2)}V_{nk}(t_1)V_{km}(t_2) \\
+ i \int_0^T dt_1 Q_{mn}(t_1) \right). \tag{72}
\]
where \( E_{nk} \equiv E_n - E_k, V_{nk}(t_1) \equiv \langle n|V(t_1)|k\rangle \), etc, and higher order terms have been neglected. Using the shorthand \( e^{i\gamma(n)} \approx 1 - i\lambda v_1 - \lambda^2 v_2 \) for the rhs of (72), we find for the residual phase
\[
\gamma(n) \approx -\lambda v_1 - \frac{i}{2} \lambda^2 (v_1^2 - 2v_2). \tag{73}
\]

5.3. Extracting the geometric phase

The first-order term in the above expression, equation (73), is the dynamical phase factor associated with the first-order correction to the energy. It is also clear that the quadratic term in the same expression contains information about the geometric phase associated with one cycle of the perturbation, but also receives contributions from the square of the first order, as well
as the second order, energy corrections. Our task is to disentangle these contributions, and our approach will be to consider a cyclic variation of the parameters of frequency $\omega$, determine the $\omega$-independent part of (73) and identify it with the geometrical phase accumulated in a period. More concretely, consider the case of two small, $O(\lambda)$, parameters $\xi$ and $\zeta$, and a first-order perturbation Hamiltonian $V(t)$ that depends not only on their values but also on those of their time derivatives,

$$\lambda V = \eta (\xi H^\xi + \dot{\xi} H^\xi + \xi H^\xi + \dot{\xi} H^\xi),$$

(74)

where $H^\xi$, $\dot{H}^\xi$, etc are assumed to be time-independent operators. Consider further the most general form of a second-order perturbation $Q(t)$ which, similarly, depends on $\xi$, $\zeta$, $\dot{\xi}$, $\dot{\zeta}$,

$$\lambda^2 Q(t) = \eta^2 (\xi^2 H^{2\xi} + \dot{\xi} \dot{\xi} H^{\dot{\xi}} + \dot{\xi} \xi H^{\dot{\xi}} + \dot{\xi} \dot{\xi} H^{\dot{\xi}} + \cdots),$$

(75)

where, initially, all ten possible terms are included. However, the antisymmetry of $K_{\xi\zeta}$ under the exchange $\xi \leftrightarrow \zeta$ implies that only the combination $H^{\xi\zeta} - H^{\zeta\xi}$ can possibly be involved in the calculation—we will let the algebra confirm this. The system now is driven around a circle in the $\xi$–$\zeta$ plane, $\xi = \lambda (\cos \omega t - 1)$, $\dot{\zeta} = \lambda \sin \omega t$, with $\omega \ll E_{nk}$, for all $k \neq n$, as required by adiabaticity. Note that the condition $V(0) = V(T) = 0$ assumed above is satisfied in the limit $\omega \to 0$.

With the time dependence of $V$ and $Q$ given, $\gamma^{(n)}$ in (73) may be expanded in powers of $\omega$, and its $\omega$-independent part $\gamma^{(0)}_0$ may be identified, giving for the curvature $K^{(n)}_{\xi\zeta} = \gamma^{(n)}_0 / (\pi \lambda^2)$

$$\eta^{-2} K^{(n)}_{\xi\zeta} = \frac{i}{\lambda^2} \sum_{k \neq n} E_{nk}^{-1} (H^{\xi}_{nk} H^{\zeta}_{kn} - H^{\xi}_{nk} H^{\zeta}_{kn})$$

$$+ \sum_{k \neq n} E_{nk}^{-1} (H^{\xi}_{nk} H^{\zeta}_{kn} + H^{\xi}_{nk} H^{\zeta}_{kn} - H^{\xi}_{nk} H^{\zeta}_{kn} - H^{\xi}_{nk} H^{\zeta}_{kn})$$

$$+ (H^{\xi\zeta} - H^{\zeta\xi})_{nn},$$

(76)

where $T = 2\pi / \omega$ was used and $K^{(n)}_{\xi\zeta}$ was obtained by dividing $\gamma^{(n)}_0$ by the area of the circle traced out in parameter space.

5.4. Discussion

Although the two expressions we have obtained for Berry’s curvature, equations (62) and (76), have in common most of their terms, we have not been able to show that, in general, they coincide. The reason why we have stopped short of proving their equivalence is that, while for (62), first-order corrections to the wavefunctions are sufficient for computing the second-order geometrical phase, this is not the case in the adapted frame calculation. There, the explicit form of (75) is needed, and that involves a lengthy expression with several dozen terms. Having said that, it is conceivable, and, in fact, we tend to believe that, some general analytical argument, that avoids brute force calculations, should exist, proving the above equivalence—we are currently working on this. Meanwhile, we emphasize that our first result for the geometrical phase, equation (62), has been derived using standard techniques, which are applicable to the problem at hand. Our subsequent attempt to repeat the calculation in the adapted frame stemmed from pure curiosity, as to how a comoving adapted observer would describe the situation. This is not just luxury; while in the present problem, the Cartesian description is available, in other situations we have in mind, e.g., geometric phases in the presence of gravitational waves, using adapted frames might be the only reasonable option.

The fact that the corresponding condition for $Q(t)$ is not satisfied has no repercussions as it affects higher orders in the deformation parameter.
Thus, we seized the opportunity to elucidate the structure of the calculation in the adapted frame, so as to gain experience for future investigations. Still, a couple of remarks are due concerning subtle points of our approach\(^8\). First, we would like to comment on the appearance of two Hamiltonians in our approach, equations (18) and (66), in conjunction with the adiabatic approximation we use. In both calculations we carry out, we assume that the particle remains in an instantaneous eigenstate of the first Hamiltonian, since it started out in one, at \(t = 0\). In particular, then, given that at \(t = 0\) the second Hamiltonian does not coincide with the first, due to the presence of the velocity terms, the evolution of the system is not adiabatic w.r.t. the second Hamiltonian. Another point worth mentioning is the \textit{a priori} possibility of the dynamical factor contributing to the geometric phase, a fact alluded to earlier (see also, in this context, [24] and references therein). We have been careful to avoid missing any such contribution. In the first calculation, no such danger exists, as the Hamiltonian lacks velocity-dependent terms. In the second calculation, where the danger is real, we have not followed the standard recipe of Berry, avoiding the dubious, in this case, splitting of the total phase into dynamical and geometric parts. Rather, we compute the time evolution with the adapted frame Hamiltonian, project to the initial state to find the total phase, and identify, in the end, its geometric, \(\omega\)-independent part. Finally, we would like to draw the reader’s attention to the fact that explicit calculations, performed in the example of harmonic deformations of a circle, give identical results by either method (see [5] and [6]). Although far from conclusive, this is further evidence that our expectations about the ultimate agreement between (62) and (75) might be well founded.

6. Summary and concluding remarks

We have studied in detail the accumulation of the geometric phase by particles constrained to move along space curves, when the latter are deformed cyclically. We furnish two descriptions of the same physical situation, one from the point of view of an inertial observer, using a Cartesian coordinate system, and the other from a noninertial one, following the curve, tracing, in each case, the exact origin of the various contributions to the phase. Our general formulas, equations (62) and (76), respectively, although quite similar in structure, differ, without this implying that they have been shown to disagree.

Geometric phases can be detected by interference, for example, by preparing initially the system in a superposition of eigenstates of the Hamiltonian, each of which acquires a different phase. Then, after a cycle, the associated probability distribution will change, in general, and this latter fact ought to be agreed upon by all observers. We expect therefore that further work, along the lines alluded to above, might remove the above apparent (we believe) discrepancy.

An obvious direction for further work along the lines presented here is the study of degenerate states, and their ensuing mixing in the manner of Wilzeck–Zee. A cyclic deformation of the curve, in that case, would give rise to a unitary mixing of the degenerate states—possible connections with topological quantum computing might prove interesting.

Another possibility, pointed out by a referee, might be a reformulation of the problem, based on the surface swept out by the wire as it is being deformed. Since the same 3D surface may be swept out by completely different deformations of the wire, we believe that an appropriate starting point for this approach would involve the world tube swept out in (Galilean) spacetime. Connections with the ideas in [23], involving horizontal lifts of surfaces, might in fact help bridge the two descriptions we have pursued here.

\(^8\) We thank the anonymous referee for insisting on clarifying some of the points mentioned here.
Regarding the possible experimental verification of these findings, we remark that the cumulative nature of the effect over time provides enhanced sensitivity in detecting minute mechanical deformations. A second method that can be used to boost observability is the employment of a large number of particles, in some sort of condensate. The total geometrical phase, in this case, is proportional to the number of particles, and if the condensate can be coupled to, e.g., a single spin, then the accumulated phase of the system could show up in changes in the spin’s ‘orientation’. An interesting arena of applications might be the detection of gravitational waves, through the minute deformations they produce to physical wires—a careful study would be necessary to verify if geometric phases are also induced in this case.

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