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GENERALIZED QUANTUM RELATIVISTIC KINEMATICS: A STABILITY POINT OF VIEW

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We apply Lie algebra deformation theory to the problem of identifying the stable form of the quantum relativistic kinematical algebra. As a warm up, given Galileo's conception of spacetime as input, some modest computer code we wrote zeroes in on the Poincaré-plus-Heisenberg algebra in about a minute. Further ahead, along the same path, lies a three-dimensional deformation space, with an instability double cone through its origin. We give physical as well as geometrical arguments supporting our view that moment, rather than position operators, should enter as generators in the Lie algebra. With this identification, the deformation parameters give rise to invariant length and mass scales. Moreover, standard quantum relativistic kinematics of massive, spinless particles corresponds to non-commuting moment operators, a purely quantum effect that bears no relation to spacetime non-commutativity, in sharp contrast to earlier interpretations.

Keywords: Poincaré algebra deformations; deformed special relativity; non-commutative spacetime; Heisenberg algebra; invariant length scale; invariant mass scale.

1. Introduction

A prevailing theme of the last decade or so in physics has been the search for an algebraic signature of quantum gravity. Lorentz symmetry violation, spacetime non-commutativity and modified dispersion relations, among other novelties, have been proposed as signals our antennas should be tuned for, in the search for a scheme where quantum objects could be heavy too. Even before that, physicists exasperated by the darker side of quantum field theory, sought their way out of the maze of infinities in the form of a spacetime granularity that would exorcise, the hope was, their ultraviolet nemeses.

The usual suspect in many of these endeavors has been the nature of spacetime, typically codified in a kinematical algebra. Accordingly, the above search has often focused on the possible deformations of the Lie algebra $\mathcal{G}_{\mathrm{PH}}$, i.e., the Poincaré algebra, extended by the inclusion of the position operators and the Heisenberg commutation relations, or one of its subalgebras. These attempts can be roughly divided into three categories, based on the mathematical framework of their approach (or the absence thereof). The first category comprises deformations of Lie type, where the commutators of the generators are linear functions of the same. There exists a well-developed mathematical formalism to deal systematically with such deformations, which has not always been used by physicists. We comment more extensively on this type of deformations below. In category number two enter quantum group type deformations, which are generalizations of the classical group concept and form particular examples of Hopf algebras with a universal R-matrix $\mathcal{R}$, that solves the (universal) quantum Yang–Baxter equation.[a] The linearity of the Lie case is lost,[b] but the construction of the algebra is canonical, given an $\mathcal{R}$ with suitable properties. Extensive work in the eighties and nineties has provided a solid mathematical background for these deformations, with applications overflowing to an impressive list of fields. Finally, recent years have seen a plethora of articles loosely classified under the generic misnomer[1] "Doubly Special Relativity" (see, e.g., Ref. 2), which form the third category. A common feature among them, and in some sense the defining one, is that the commutators of the algebra are given by general analytic functions of the generators.[c] To our knowledge, there is no well-defined mathematical framework guaranteeing the self-consistency of these deformations, not to mention their physical applicability, partly because they are not complete, e.g., the fate of the spacetime sector is often left unclear. Subsequent work[20] showed that endowing the above deformations with considerable more (Hopf) structure, results in their identification with particular forms of the κ-Poincaré Hopf algebra, proposed about a decade ago[24] (see also Ref. 25). Both of the last two categories suffer from serious physical problems in the many-particle sector, e.g., in a consistent definition of such basic quantities like the total momentum of a system of particles.

In this paper we deal with Lie-type deformations of standard quantum relativistic kinematics. We undertook this project with three main goals in mind:

(1) Emphasize the Lie algebra stability point of view and present in an accessible form the relevant mathematical apparatus, along the lines of Ref. 27, which motivated the present work.
(2) Apply the formalism to the problem at hand to obtain a complete, detailed map of the deformation territory in the vicinity of $\mathcal{G}_{\rm PH}$.
(3) Interpret physically the generators of the algebra and investigate the nature of the deformations.

[a]Not all Hopf-type deformations are known to possess a universal R-matrix, and some are known not to.
[b]For a class of such algebras, an appropriate deformation of the concept of commutator restores linearity (see, e.g., Ref. 33).
[c]The introduction of additional invariant scales cannot be considered as a defining characteristic of these deformations since, as is well-known, and as we are about to see, such scales are also introduced by the Lie-type deformations.

The structure of the paper was conceived accordingly, with each of the subsequent three sections dealing with one of the above goals. In Section 2 we give a self-contained review of the standard Lie algebra deformation theory and explain why stable structures are more likely to prove useful in physical applications than unstable ones. The section ends with a relatively detailed example, the passage from Galilean to relativistic kinematics, illustrating the use of the formalism, as well as the (alas, *a posteriori*) predictive power of the stability point of view. Section 3 contains a detailed analysis of the options available in deforming $\mathcal{G}_{\mathrm{PH}}$. We take as our starting point classical ($\hbar = 0$) relativistic kinematics (an unstable algebra) and, with the help of some computer code we wrote, explore the various paths that lead to stable algebras. We find that there is essentially one path, its first stop introducing Heisenberg's relations. Thus, given Galileo's conception of spacetime as input, our program zeroes in on the Poincaré-plus-Heisenberg algebra $\mathcal{G}_{\mathrm{PH}}$ in about a minute. We find this motivating enough to inquire about what lies further ahead. Following this path to its end, we find ourselves in a three-dimensional deformation space of stable Lie algebras, with a double instability cone through its origin. The section ends with a description of relations between our work and earlier treatments in the literature. This concludes the mathematical part of the paper — inferences about physics will have to wait the physical identification of the generators, which we undertake in Section 4. There, we argue that the position operators do not have the right properties to serve as Lie algebra generators. In doing so, we are in disagreement with all previous works. Retracing the steps that lead to the definition of the relativistic center-of-momentum concept for a system of particles, we come to the conclusion that the appropriate generators, in the case of a massive, spinless particle, are the moment operators, given essentially by the positions rescaled by the mass operator for the particle. This shows that the algebra $\mathcal{G}_{\mathrm{QR}}$ of standard quantum relativistic kinematics, differs from $\mathcal{G}_{\mathrm{PH}}$ and, in the above case, lies on the instability cone. Furthermore, from the algebraic point of view, there is a single deformation direction introducing non-commutativity among the momenta. Section 5 comments on the findings and outlines directions for future work.

2. Lie Algebra Deformations and the Concept of Stability

In this section we summarize the elements of standard Lie algebra deformation theory that will be of use in the rest of the paper. Our exposition follows mostly the original source for this material, Refs. 29 and 30 as well as Ref. 27. Section 2.5 follows Ref. 7, echoing ideas originating in the Batalin–Vilkovisky quantization (see, e.g., Ref. 37), the deformation-theoretic aspects of the latter being first pointed out in Ref. 3 (see also Ref. 15). Background information on Lie algebra cohomology can be extracted (not without some effort) from Refs. 6 and 16. The foundations of deformation theory are laid out in the classic Ref. 11, while plenty of newer material is contained in the book-length Ref. 12. An elegant generalization to bialgebra deformations was given in Ref. 13, with still further generalizations to Drinfeld algebras, and much more, appearing in Ref. 34 — this latter reference also contains a rather comprehensive bibliography. An exposition of related material, with physical applications in mind, can be found in Ref. 10.

2.1. *Lie products*

We deal throughout with finite-dimensional real Lie algebras. These are built on a (finite-dimensional) real vector space V, by defining a bilinear antisymmetric *Lie product* map $\mu: V \times V \to V$ that satisfies the Jacobi identity,

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) + \mu(y, \mu(x, z)) \,. \tag{1}$$

This is usually written as a cyclic sum, a form that, in the case at hand, obscures its content. To clarify the latter, take as an example the case where x is a Lorentz group generator, $J_{\mu\nu}$, and y, z are other generators carrying Lorentz indices, say, Y_ρ, Z_σ respectively. Suppose $\mu(y, z) = \mu(Y_\rho, Z_\sigma) = W$. Substituting this in the l.h.s. of (1), one finds that the Jacobi identity requires that the transformation properties of W under the Lorentz group are derived solely from those of Y_ρ, Z_σ, i.e., in this case, W ought to transform as a second-rank covariant tensor. Another way of saying this is that μ itself is a Lorentz scalar, an observation that we use later on.

Given a basis $\{T_A\}$, $A = 1, \ldots, n$ of V, the product μ is specified by giving all vectors $\mu(T_A, T_B)$, $1 \leq A < B \leq n$. The coordinates of these vectors in the basis are, up to a factor of i, the *structure constants* ${f_{AB}}^C$ of the algebra,

$$[T_A, T_B] \equiv i\mu(T_A, T_B) = i{f_{AB}}^C T_C \,, \tag{2}$$

which are antisymmetric in the lower two indices (a sum over repeated indices is implied). In the above equation we follow the standard physics practice of expressing the (non-associative) Lie product as the commutator $[\cdot\,,\cdot]$ with respect to an associative operator product, as well as the inclusion of an imaginary unit, related to the Hermiticity of the generators. In terms of the structure constants, the Jacobi identity becomes

$${f_{AR}}^S {f_{BC}}^R + {f_{BR}}^S {f_{CA}}^R + {f_{CR}}^S {f_{AB}}^R = 0 \,. \tag{3}$$

Relaxing for the moment this latter constraint, i.e., taking into account only the antisymmetry in the lower two indices, one is left with $N(n) = n^2(n-1)/2$ arbitrary constants ${f_{AB}}^C$, $A < B$. Consider now the space $\mathbb{R}^N$, with each of the f's ranging along an axis. For each value of (A, B, C, S), (3) describes a quadratic hypersurface in this space. The intersection of these hypersurfaces is the space $\mathcal{L}_n$ of all possible n-dimensional Lie algebras — we sketch it as a surface in Fig. 1. Referring to this figure, consider the point P of $\mathcal{L}_n$ — it corresponds to the Lie algebra $\mathcal{G}_P$, whose structure constants are given by the coordinates of P. Under a linear redefinition of the generators via a $\mathrm{GL}(n)$ matrix M,

$$T'_A = {M_A}^B T_B \,, \tag{4}$$

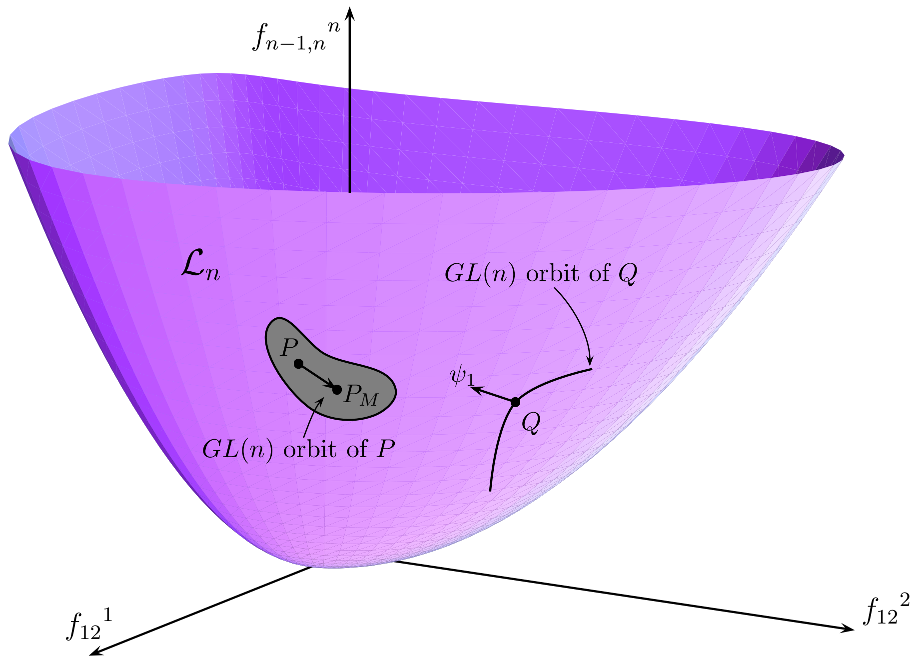

Figure 1. The space $\mathcal{L}_n$ of n-dimensional Lie algebras (sketch). P is surrounded by equivalent points and hence, $\mathcal{G}_P \sim \mathcal{G}_{P_M}$, for all P_M sufficiently close to P. In contrast, in the tangent space of $\mathcal{L}_n$ at Q, there are directions that lead outside of the $\mathrm{GL}(n,\mathbb{R})$ orbit $\mathrm{Orb}(Q)$. Q will move along these directions when ψ_1 in (6) is a non-trivial element of $H^2(\mathcal{G}_Q)$. Notice that, for all n, the surface passes through the origin, which corresponds to the n-generator Abelian algebra.

the structure constants transform as

$$f'_{AB}{}^C = M_A{}^R M_B{}^S (M^{-1})_U{}^C f_{RS}{}^U \,, \tag{5}$$

and P moves to P_M. Clearly, no new physics is to be expected from such a redefinition, $\mathcal{G}_P$ and $\mathcal{G}_{P_M}$ being isomorphic. What we are really interested in then, from a physical point of view, is not $\mathcal{L}_n$ itself, but, rather, the space of equivalence classes into which $\mathcal{L}_n$ splits under the above action of $\mathrm{GL}(n)$, each class being the $\mathrm{GL}(n)$ orbit $\mathrm{Orb}(P)$ of any point P in the class.[d] The crucial observation to be made here is that there exist two types of points in $\mathcal{L}_n$: those that are completely surrounded by equivalent points (corresponding to isomorphic algebras) and those whose neighborhoods[e] include non-equivalent points, sketched as P and Q respectively in Fig. 1. Any infinitesimal perturbation of the structure constants of $\mathcal{G}_P$ will necessarily lead to an isomorphic Lie algebra — the orbit $\mathrm{Orb}(P)$ is open in $\mathcal{L}_n$. We call $\mathcal{G}_P$ *stable* or *rigid*. On the other hand, there exist infinitesimal

[d]Notice that if one changes the structure constants from f to f', as above, *without* changing the generators, one *does* obtain new physics.
[e]$\mathcal{L}_n$ inherits the natural topology of the structure constants, i.e., that of the ambient $\mathbb{R}^N$.

perturbations of $\mathcal{G}_Q$ that lead outside of $\mathrm{Orb}(Q)$ and, hence, to non-isomorphic algebras — we call $\mathcal{G}_Q$ *unstable.*

In physical applications, structure constants are often given by experimentally determined fundamental constants of the theory. The experimental errors involved render the position of the corresponding algebra in $\mathcal{L}_n$ uncertain. If the algebra employed is unstable, the physical predictions of the theory become ill defined, as they depend critically on the exact value of the structure constants, which is not known. Additionally, new measurements or improved data analysis, may move the algebra to a new position. If the algebra is stable, the physical theory based on it will maintain its qualitative validity. We conclude that stable algebras give rise to robust physics.

2.2. *Deformations and H^2*

Given a Lie algebra $\mathcal{G}_0 = (V, \mu_0)$, i.e., the Lie product of X, $Y \in V$ is supplied by $\mu_0(X,Y) \equiv [X,Y]_0$. A *one-parameter (formal) deformation* of $\mathcal{G}_0$ is given by the *deformed commutator*

$$[X,Y]_t = [X,Y]_0 + \sum_{m=1}^{\infty} \psi_m(X,Y) t^m \,, \tag{6}$$

where t is a formal parameter. The corresponding t-dependent structure constants,

$$[T_A, T_B]_t = i f^{t}_{AB}{}^{C} T_C \,, \tag{7}$$

define a curve P_t in $\mathcal{L}^n$, which passes through P_0 (corresponding to $\mathcal{G}_0$) at $t = 0$. The l.h.s. of (6) is bilinear and antisymmetric, hence the ψ_m on the r.h.s. are $\mathcal{G}$-valued, bilinear antisymmetric maps

$$\psi_m : V \times V \to V \,, \qquad \psi_m(X,Y) = -\psi_m(Y,X) \,. \tag{8}$$

We will call such maps *2-cochains* (over V), extending the definition in the natural way (i.e., via p-linearity and total antisymmetry) to *p-cochains* $\psi^{(p)}$, which accept p arguments.[f] The vector space of p-cochains over V will be denoted by $C^p(V)$. Notice that the 1-cochains are simply linear maps from V to V, the antisymmetry requirement being meaningless in this case. Also, the space of 0-cochains is V itself. Next, for an arbitrary Lie product μ, we define a *coboundary operator* s_μ, which maps p-cochains to $(p+1)$-cochains, $s_\mu \colon C^p \to C^{p+1}$, according to

$$\begin{aligned} s_\mu \triangleright \psi^{(p)}&(T_{A_0}, \ldots, T_{A_p}) \\ &= \sum_{r=0}^{p} (-1)^r \mu\big(T_{A_r}, \psi^{(p)}(T_{A_1}, \ldots, \hat{T}_{A_r}, \ldots, T_{A_p})\big) \\ &\quad + \sum_{r<s} (-1)^{r+s} \psi^{(p)}(\mu(T_{A_r}, T_{A_s}), T_{A_0}, \ldots, \hat{T}_{A_r}, \ldots, \hat{T}_{A_s}, \ldots, T_{A_p}) \end{aligned} \tag{9}$$

[f] When the order p of a cochain ψ needs to be emphasized, we will write $\psi^{(p)}$.

(hats denote omitted terms). For example, for $\phi \in C^1$ and $\psi \in C^2$,

$$\begin{aligned} s_\mu \triangleright \phi(A_1, A_2) &= [A_1, \phi(A_2)] - [A_2, \phi(A_1)] - \phi([A_1, A_2])\,, \\ s \triangleright \psi(A_1, A_2, A_3) &= [A_1, \psi(A_2, A_3)] - [A_2, \psi(A_1, A_3)] + [A_3, \psi(A_1, A_2)] \\ &\quad - \psi([A_1, A_2], A_3) + \psi([A_1, A_3], A_2) - \psi([A_2, A_3], A_1)\,, \end{aligned} \tag{10}$$

where $\mu(X,Y) = [X,Y]$. It can be shown that $s_\mu^2 = 0$, a result that relies on the Jacobi identity that μ satisfies — a compact proof is given in Section 2.3. The relevance of s_μ to the problem at hand becomes evident when one imposes the Jacobi identity on the deformed commutator in (6). Writing out this identity and evaluating its t-derivative at $t = 0$, one finds that ψ_1 must satisfy $s_{\mu_0} \triangleright \psi_1 = 0$. We call a p-cochain $\psi^{(p)}$ annihilated by s_μ, $s_\mu \triangleright \psi^{(p)} = 0$, a *$p$-cocycle*, and denote the vector space of p-cocycles by $Z^p(V, s_\mu)$. What we have found above is that *infinitesimal deformations of Lie algebras are generated by 2-cocycles*, the converse being also true.

It remains to determine which of the above infinitesimal deformations lead to isomorphic Lie algebras. As mentioned already, isomorphic Lie algebras result from a linear redefinition of the generators with some invertible matrix $M \in \mathrm{GL}(n, \mathbb{R})$ (see (4)). In the case of a deformation, $M = M_t$ is t-dependent, with $M_0 = I_n$ (the unit $n \times n$ matrix). Then the deformed, t-dependent commutator, is given by

$$[X, Y]_t = M_t^{-1}[M_t X, M_t Y]_0\,, \tag{11}$$

for any X, Y in $\mathcal{G}_0$. Taking M_t in a neighborhood of the identity, $M_t = I_n + tQ$, with t small, one readily computes the corresponding first-order (in t) change to the commutator,

$$\begin{aligned} [X, Y]_t &= M_t^{-1}[M_t X, M_t Y]_0 \\ &= (I_n - tQ)[(I_n + tQ)X, (I_n + tQ)Y]_0 \\ &= [X, Y]_0 + t([X, QY]_0 - [Y, QX]_0 - Q[X, Y]_0) + \mathcal{O}(t^2)\,. \end{aligned} \tag{12}$$

But the linear map $Q\colon X = X^A T_A \mapsto QX = X^R Q_R{}^S T_S$ is, as mentioned earlier, a 1-cochain. Comparing the $\mathcal{O}(t)$-term in the r.h.s. of (12) with the first of the examples in (10) shows that the $\mathcal{O}(t)$-change in the commutator, i.e., the 2-cochain ψ_1 in (6), is given by

$$\psi_1 = s_{\mu_0} \triangleright Q\,. \tag{13}$$

We call a p-cochain $\psi^{(p)}$ that is in the image of s_μ, $\psi^{(p)} = s_\mu \triangleright \phi^{(p-1)}$, a *trivial p-cocycle*, or, a *p-coboundary*. The vector space of p-coboundaries will be denoted by $B^p(V, s_\mu)$. Since $s_\mu^2 = 0$, all coboundaries are cocycles, $B^p \subseteq Z^p$. What the above result shows is that *infinitesimal deformations of $\mathcal{G}_0$ towards isomorphic Lie algebras are generated by 2-coboundaries.*

Conversely, assume that for the deformed commutator as in (6), there exists a linear map (1-cochain) $\phi_1 : \mathcal{G}_0 \to \mathcal{G}_0$, such that the ψ_1 appearing in the r.h.s. of that equation is given by $\psi_1 = s_{\mu_0} \triangleright \phi_1$. Consider now a linear redefinition of the generators by the matrix $M_1 = e^{-t\phi_1}$ and compute the new t-commutator $[X,Y]'_t$. The result is given by (12), with the substitutions $[X,Y]_t \to [X,Y]'_t$ and $[X,Y]_0 \to [X,Y]_t$,

$$[X,Y]'_t = [X,Y]_t - t\,\underbrace{s_{\mu_0} \triangleright \phi_1}_{\psi_1}(X,Y) + \mathcal{O}(t^2)\,. \tag{14}$$

Using (6) to expand the r.h.s. in powers of t, we see that the term linear in t in $[X,Y]'_t$ cancels. Repeating the procedure one may eliminate one by one all powers of t, thus bringing the original t-commutator in coincidence with the undeformed one $[X,Y]_0$, using nothing more than successive linear redefinitions of the generators. We conclude that the two commutators define isomorphic Lie algebras, the matrix giving the isomorphism being $M = \cdots M_2 M_1$, with $M_m = e^{-t\phi_m}$ and $s_{\mu_0} \triangleright \phi_m = \psi_m$.

We may summarize the contents of this section in the following geometrical picture: the tangent space[g] $T_{P_0}\mathcal{L}_n$ to $\mathcal{L}_n$ at P_0 is (isomorphic to) Z^2, the space of 2-cocycles. The subspace of $T_{P_0}\mathcal{L}_n$ leading to isomorphic Lie algebras, i.e., the tangent space to the $\mathrm{GL}(n)$-orbit $\mathrm{Orb}(P_0)$ is B^2, the space of 2-coboundaries. To close the familiar circle of definitions, we define the quotient space $H^p \equiv Z^p/B^p$, in which two cocycles are identified if they differ by a coboundary, as the *pth cohomology group*[h] of $\mathcal{G}_0$. The non-trivial elements of H^2 (if any) correspond to directions in $T_{P_0}\mathcal{L}_n$ that lead to Lie algebras infinitesimally close to $\mathcal{G}_0$ but non-isomorphic to it. A sufficient condition then for the stability of $\mathcal{G}_0$ is the vanishing of its second cohomology group $H^2(\mathcal{G}_0)$. Whitehead's lemma states that this condition is satisfied by all semisimple Lie algebras[18] — we conclude that *semisimple Lie algebras are stable.* It is worth pointing out that the above is not a necessary condition. As explained in Section 2.4, although a non-trivial 2-cocycle may exist, obstructions originating in $H^3(\mathcal{G}_0)$ can render it non-integrable, in which case the corresponding *finite* non-trivial deformation does not exist. Concrete examples of stable Lie algebras with non-trivial H^2 have been constructed, typically as semidirect products. For example (see Ref. 31), denote by S the simple three-dimensional Lie algebra over $\mathbb{C}$ and by ρ_n the irreducible representation of weight n of S on $W \equiv \mathbb{C}^{2n+1}$. The semidirect product $L_n = W \rtimes_{\rho_n} S$, for $n > 5$ and odd, is a stable Lie algebra, while its second cohomology group is non-trivial. To deal with such cases, non-cohomological approaches have been developed, relying on techniques of non-standard analysis. A classification algorithm for stable Lie algebras exists, relying on a theorem that such algebras possess a standard non-zero generator whose adjoint representation

[g] We are assuming here that P_0 is not a singular point of $\mathcal{L}_n$ — if that is not the case one may instead conclude that the Zariski tangent space to $\mathcal{L}_n$ at P_0 is Z^2 (see Refs. 30 and 14, p. 317).
[h] Z^p, B^p, H^p are all Abelian groups with the group composition given by addition.

is diagonalizable. Although tedious, the algorithm permits, in principle, the classification of all stable Lie algebras, in any dimension — for more details we refer the reader to Refs. 4 and 14.

2.3. *The $\bar{\wedge}$ product*

It turns out that calculations involving expressions like (1), or (10), simplify considerably when a particular product, the subject of this section, is introduced among p-cochains.[30]

Given a vector space V, put $\mathrm{Alt}^p(V) = C^{p+1}(V)$, $p \geq -1$. Then for $\alpha \in \mathrm{Alt}^m(V)$, $\beta \in \mathrm{Alt}^n(V)$, define the product $\alpha \,\bar{\wedge}\, \beta \in \mathrm{Alt}^{m+n}(V)$ by

$$\alpha \,\bar{\wedge}\, \beta(X_0, \ldots, X_{m+n}) = \sum_{\sigma} \mathrm{sgn}(\sigma)\alpha(\beta(X_{\sigma(0)}, \ldots, X_{\sigma(n)}), X_{\sigma(n+1)}, \ldots, X_{\sigma(m+n)}) \,, \tag{15}$$

where σ ranges over all permutations such that $\sigma(0) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(m+n)$ (these are known as *riffle shuffles with cut at* $n+1$). When both α and β are 2-cochains, as will often be the case, the above formula reduces to

$$(\alpha \,\bar{\wedge}\, \beta)_{ABC}{}^{T} = \alpha_{RA}{}^{T}\beta_{BC}{}^{R} + \alpha_{RB}{}^{T}\beta_{CA}{}^{R} + \alpha_{RC}{}^{T}\beta_{AB}{}^{R} \,, \tag{16}$$

where, for a p-cochain $\psi^{(p)}$,

$$\psi^{(p)}(T_{A_1}, \ldots, T_{A_p}) = \psi_{A_1 \cdots A_p}{}^{B} T_B \,. \tag{17}$$

Notice that $\bar{\wedge}$ is non-associative, but satisfies instead

$$(\gamma \,\bar{\wedge}\, \alpha) \,\bar{\wedge}\, \beta - \gamma \,\bar{\wedge}\, (\alpha \,\bar{\wedge}\, \beta) = (-1)^{mn}((\gamma \,\bar{\wedge}\, \beta) \,\bar{\wedge}\, \alpha - \gamma \,\bar{\wedge}\, (\beta \,\bar{\wedge}\, \alpha)) \tag{18}$$

(the *commutative-associative law*). The (*graded*) *commutator* of α, β is defined as

$$[\![\alpha, \beta]\!] = \alpha \,\bar{\wedge}\, \beta - (-1)^{mn}\beta \,\bar{\wedge}\, \alpha \,. \tag{19}$$

Consider now a Lie algebra $\mathcal{G} = (V, \mu)$, $\mu \in C^2(V) = \mathrm{Alt}^1(V)$. It is easy to see that the Jacobi identity for μ, Eq. (1), can be put in the form

$$\mu \,\bar{\wedge}\, \mu = \frac{1}{2}[\![\mu, \mu]\!] = 0 \tag{20}$$

(the first equality is an immediate consequence of (19)). Furthermore, the action of the coboundary operator s_μ on an arbitrary $(p+1)$-cochain $\psi \in \mathrm{Alt}^p(V)$ is given by

$$s_\mu \triangleright \psi = (-1)^p[\![\mu, \psi]\!] \equiv (-1)^p D_\mu \psi \,, \tag{21}$$

i.e., s_μ is equal, up to a sign depending on the order of the cochain it acts on, to the operator $D_\mu \equiv [\![\mu, \cdot\,]\!]$. Thus, all operations introduced in earlier sections can be expressed in terms of the $\bar{\wedge}$ product.

It can be shown that the graded commutator of (19) satisfies a graded Jacobi identity,

$$(-1)^{mp}[\![\alpha,[\![\beta,\gamma]\!]]\!]+(-1)^{nm}[\![\beta,[\![\gamma,\alpha]\!]]\!]+(-1)^{pn}[\![\gamma,[\![\alpha,\beta]\!]]\!]=0\,, \tag{22}$$

where $\alpha\in\mathrm{Alt}^m(V)$, $\beta\in\mathrm{Alt}^n(V)$ and $\gamma\in\mathrm{Alt}^p(V)$. This property, together with bilinearity and graded antisymmetry, implies that $\mathrm{Alt}(V)\equiv\bigoplus_n\mathrm{Alt}^n(V)$ is a graded Lie algebra.

We derive now a number of interesting results, illustrating along the way the efficiency afforded by the formalism introduced in this section. First, the proof of $s_\mu\circ s_\mu=0$ may be given in a simplified form. Up to an irrelevant sign, it translates into $D_\mu\circ D_\mu=0$ and, for an arbitrary $\alpha\in\mathrm{Alt}(V)$,

$$D_\mu\circ D_\mu\alpha=[\![\mu,[\![\mu,\alpha]\!]]\!]=\frac{1}{2}[\![[\![\mu,\mu]\!],\alpha]\!]=0\,, \tag{23}$$

where the second equality follows from the graded Jacobi identity for $[\![\cdot\,,\cdot]\!]$, Eq. (22), and the last one from the Jacobi identity for μ, Eq. (20). Second, the equation for *finite* deformations may be derived easily. If μ is a Lie product, $\mu'=\mu+\phi$ will also be one if $[\![\mu',\mu']\!]=0$, from which one gets immediately the *deformation equation*

$$D_\mu\phi+\frac{1}{2}\,[\![\phi,\phi]\!]=0\,, \tag{24}$$

which reduces to the cocycle condition for infinitesimal ϕ. Third, Eq. (22) implies that D_μ is a graded derivation in $\mathrm{Alt}(V)$, i.e.,

$$D_\mu[\![\alpha,\beta]\!]=[\![D_\mu\alpha,\beta]\!]+(-1)^m[\![\alpha,D_\mu\beta]\!]\,, \tag{25}$$

where $\alpha\in\mathrm{Alt}^m(V)$ and $\beta\in\mathrm{Alt}(V)$. One may then conclude that if α, β are cocycles, α, $\beta\in Z(\mathrm{Alt}(V),D_\mu)$, then so is $[\![\alpha,\beta]\!]$, and that if, additionally, γ is a coboundary, $\gamma\in B(\mathrm{Alt}(V),D_\mu)$, then so is $[\![\alpha,\gamma]\!]$. These two facts, in turn, imply that the quotient space $H(\mathrm{Alt}(V),D_\mu)$ is itself a graded Lie algebra.

2.4. *Obstructions and H^3*

Given a Lie algebra $\mathcal{G}=(V,\mu)$ and a deformation μ_t,

$$\mu_t=\mu+\phi_t\,,\qquad \phi_t=\sum_{n=1}^{\infty}\phi_n t^n\,. \tag{26}$$

Then the deformation equation for ϕ_t, Eq. (24), implies an infinite sequence of equations for the ϕ_n, one for each power of t. The equations corresponding to t, t^2 and t^3, are[i]

$$D_\mu\phi_1=0\,, \tag{27}$$

$$D_\mu\phi_2=-\frac{1}{2}[\![\phi_1,\phi_1]\!]\,, \tag{28}$$

$$D_\mu\phi_3=-[\![\phi_1,\phi_2]\!]\,. \tag{29}$$

[i]Notice that all the ϕ_n are 2-cochains, so that $[\![\phi_m,\phi_n]\!]=[\![\phi_n,\phi_m]\!]=\phi_m\,\bar{\wedge}\,\phi_n+\phi_n\,\bar{\wedge}\,\phi_m$.

If ϕ_1 is a 2-cocycle, as (27) demands, then $[\![\phi_1, \phi_1]\!]$ is a 3-cocycle, since D_μ is a (graded) derivation with respect to the $[\![\cdot\,,\cdot]\!]$ product. But then, (28) demands that this 3-cocycle be a coboundary, which may not be the case if $H^3(V, D_\mu)$ is non-trivial. We see then that the existence of non-trivial 3-cocycles may render infinitesimal deformations (ϕ_1 above) non-integrable. If $[\![\phi_1, \phi_1]\!]$ is indeed a trivial 3-cocycle, so that (28) admits a solution, an obstruction may occur in the next step, i.e., in (29), and so on. It can be shown that all of these obstructions lie in H^3, so that, if H^3 is trivial, every non-trivial 2-cocycle is the first order term of some finite deformation.[30]

The following remarks will prove useful:

(1) If a non-trivial 2-cocycle ϕ also satisfies $[\![\phi, \phi]\!] = 0$, then it satisfies the deformation equation (24). In that case, the truncated deformation $\mu_t = \mu + t\phi$ is a Lie product for every t, if μ is one, regardless of the structure of H^3.

(2) If there are several non-trivial 2-cocycles ϕ_i and all their anticommutators are zero, $[\![\phi_i, \phi_j]\!] = 0$, $\forall i, j$, then an arbitrary linear combination of them also satisfies the finite deformation equation, and the space of finite deformations becomes a vector space, spanned by the ϕ_i's.

(3) In both of the above cases, infinitesimal deformations along non-trivial 2-cocycles are guaranteed to lead, as we saw earlier, to non-isomorphic algebras. This is not necessarily the case for finite deformations: the algebra $\mu + t\phi$ may become isomorphic, for particular finite values of t, to the algebra μ. Notice also that, in general, the algebras $\mu + t\phi$, for various finite values of t, may not be isomorphic among themselves. The infinitesimal version of this is that the algebras $\mu + t\phi$, for various (infinitesimal) values of t, are all isomorphic, as long as t does not change sign. $\mu + t\phi$ might well be non-isomorphic to $\mu - t\phi$, even for t infinitesimal.

Interestingly enough, these scenarios are realized in the stability analysis of the Galilean algebra, in Section 2.6, as well as in that of the PH algebra, in Section 3.

2.5. *Coboundary operator as exterior covariant derivative*

It is obvious from the definition given above, that a p-cochain can be realized as a $\mathcal{G}$-valued left invariant (LI) p-form on the group manifold G corresponding to $\mathcal{G}$, with the generators T_A now extended to LI vector fields. Denoting by $\{\Pi^A\}$ the LI 1-forms on G dual to the generators $\{T_B\}$,

$$\langle \Pi^A, T_B \rangle = \delta_B{}^A\,, \qquad (\text{with } \langle \Pi^{\mu\nu}, T_{\rho\sigma}\rangle = \delta_{\rho\sigma}{}^{\mu\nu} \equiv g_\rho{}^\mu g_\sigma{}^\nu - g_\rho{}^\nu g_\sigma{}^\mu)\,, \tag{30}$$

we write $\psi^{(p)}$ as

$$\psi^{(p)} \equiv \psi^B \otimes T_B = \frac{1}{p!}\psi_{A_1\cdots A_p}{}^B\, \Pi^{A_1}\cdots\Pi^{A_p} \otimes T_B\,. \tag{31}$$

Then the action of s given in (9) coincides with that of an exterior covariant derivative ∇,

$$\nabla(\psi^A \otimes T_A) = (d\psi^A + \Omega^A{}_B \psi^B) \otimes T_A\,, \tag{32}$$

with the connection 1-form Ω given by

$$\Omega^A{}_B = f_{RB}{}^A \Pi^R\,, \qquad (\text{i.e., } \nabla_{T_A} T_B = [T_A, T_B])\,. \tag{33}$$

The nilpotency of s follows now from the vanishing of the curvature 2-form $\Theta = d\Omega + \Omega^2$, due to the Jacobi identity, while 2-cocycles are covariantly constant $\mathcal{G}$-valued LI 2-forms (see, e.g., Ref. 7). Notice that the requirement that $s \triangleright \psi^{(2)} = 0$, with $\psi^{(2)}$ as in (31), reduces to

$$\begin{aligned} & f_{AR}{}^S \psi_{BC}{}^R + f_{BR}{}^S \psi_{CA}{}^R + f_{CR}{}^S \psi_{AB}{}^R \\ & \quad + \psi_{AR}{}^S f_{BC}{}^R + \psi_{BR}{}^S f_{CA}{}^R + \psi_{CR}{}^S f_{AB}{}^R = 0\,, \end{aligned} \tag{34}$$

which is, as expected, the first-order term, in t, of the Jacobi identity for the structure constants $f + t\psi$. The use of the differential forms language permits writing out cochains as geometrical objects, as in (31), rather than listing their components, a practice we adhere to in the following.

The point of view sketched here has been further developed, in the case of compact Lie algebras, in Ref. 38, the motivation there being the study of BRST cohomology. The appropriate coboundary operator,[j] called there the BRST operator, is realized in terms of fermionic coordinates and their dual derivatives. An involution of the algebra of the latter, made possible by the invertibility of the Killing form, gives rise to a dual object, the anti-BRST operator, and a grade-preserving Laplacian. Further generalizations, involving higher order invariant tensors of the algebra, have been explored in Ref. 8. We have developed similar techniques to deal with the non-compact case, reinstating the connection term, and used them in one of our programming approaches — we defer further details to a future publication.

2.6. *An example: The shortest path from Galileo to Einstein*

Consider the Galilean algebra $\mathcal{G}_{\mathrm{G}}$ of non-relativistic kinematics,

$$[J_a, J_b] = i\epsilon_{ab}{}^c J_c\,, \quad [J_a, K_b] = i\epsilon_{ab}{}^c K_c\,, \quad [K_a, K_b] = 0\,, \tag{35}$$

where J_a, K_a, $a = 1, 2, 3$, are the generators of rotations and boosts, respectively, and indices are raised and lowered with the unit metric. The 2-cochain μ that corresponds to this Lie product is, in the language of the preceding section,

$$\mu = \frac{1}{2}\epsilon_{ab}{}^c \Pi^a \Pi^b \otimes J_c + \epsilon_{ab}{}^c \Pi^a \Pi^{\bar{b}} \otimes K_c\,. \tag{36}$$

[j]Because of the compactness of the algebras studied in Ref. 38, the connection term in ∇ is dropped — otherwise the cohomology is trivial, as asserted by Whitehead's lemma.

We adopt here the convention that, in 1-forms, unbarred indices refer to rotations, while barred ones to boosts, so that, e.g., $\langle \Pi^{\bar{a}}, K_b \rangle = \delta_b{}^a$ (notice that bars are important in forms but make no difference in Kronecker deltas or in the summation convention). By an argument based on the observation made after Eq. (1), we conclude that only scalar (under rotations) cochains need be considered. We simplify further the notation taking advantage of the fact that, due to the limited number of generators and invariant tensors, a simple listing of the nature of the 1-forms and generators that enter in any given cochain, of up to second degree, is sufficient to reconstruct it (there is only one way to contract the indices). For example, μ above is given by

$$\mu = \chi_{\text{JJJ}} + \chi_{\text{JKK}} \,, \quad \text{where} \;\; \chi_{\text{JJJ}} \equiv \frac{1}{2}\epsilon_{ab}{}^{c}\Pi^a\Pi^b \otimes J_c \,, \quad \chi_{\text{JKK}} \equiv \epsilon_{ab}{}^{c}\Pi^a\Pi^{\bar{b}} \otimes K_c \tag{37}$$

(a factor of $1/p!$ is included whenever p 1-forms of the same type are multiplied).

We inquire now about the stability of this algebra. The most general scalar 1-cochain is given by

$$\phi = \alpha_1\phi_{\text{JJ}} + \alpha_2\phi_{\text{KJ}} + \alpha_3\phi_{\text{JK}} + \alpha_4\phi_{\text{KK}} \,, \tag{38}$$

with $\phi_{\text{JJ}} = \Pi^a \otimes J_a$, etc. Applying ∇ to obtain the most general 2-coboundary we get

$$\nabla\phi = \alpha_1(\chi_{\text{JJJ}} + \chi_{\text{JKK}}) + 2\alpha_2\chi_{\text{KKK}} + \alpha_3\chi_{\text{JJK}} \,. \tag{39}$$

On the other hand, the most general scalar 2-cochain is given by

$$\chi = \beta_1\chi_{\text{JJJ}} + \beta_2\chi_{\text{JJK}} + \beta_3\chi_{\text{JKJ}} + \beta_4\chi_{\text{JKK}} + \beta_5\chi_{\text{KKJ}} + \beta_6\chi_{\text{KKK}} \,. \tag{40}$$

We set $\nabla\chi = 0$ to obtain

$$\nabla\chi = (\beta_1 - \beta_4)\Psi_1 + \beta_3\Psi_2 = 0 \tag{41}$$

with

$$\Psi_1 = \Pi^a\Pi^b\Pi_{\bar{b}} \otimes K_{\bar{a}} \,, \quad \Psi_2 = \Pi^a\Pi^b\Pi_{\bar{a}} \otimes J_b + \Pi^{\bar{b}}\Pi^a\Pi_{\bar{a}} \otimes K_b \,. \tag{42}$$

We conclude that $\beta_1 = \beta_4$ and $\beta_3 = 0$, so that the most general 2-cocycle is given by

$$\tilde{\chi} = c_1(\chi_{\text{JJJ}} + \chi_{\text{JKK}}) + c_2\chi_{\text{JJK}} + c_3\chi_{\text{KKJ}} + c_4\chi_{\text{KKK}} \,, \tag{43}$$

with arbitrary c_i. Comparison of $\tilde{\chi}$ with $\nabla\phi$ shows that only χ_{KKJ} is a non-trivial 2-cocycle, giving for the second cohomology group

$$H^2(\mathcal{G}_{\text{G}}) = \{[0], [\chi_{\text{KKJ}}]\} \,. \tag{44}$$

Accordingly, $\mathcal{G}_{\text{G}}$ is infinitesimally unstable. By noting that $[\![\chi_{\text{KKJ}}, \chi_{\text{KKJ}}]\!] = 0$, we conclude that $\mu_t = \mu + t\chi_{\text{KKJ}}$ yields a one-parameter deformation of the algebra for finite t (see the comment at the end of Section 2.4). A look at the form of χ_{KKJ}

shows that the deformation only adds a rotation generator in the r.h.s. of the K–K commutator,

$$[K_a, K_b]_t = it\epsilon_{ab}{}^c J_c \,, \tag{45}$$

leaving the rest of the commutators intact. The Lorentz algebra, describing relativistic kinematics, sits at $t = -\frac{1}{c^2}$, where c is the velocity of light and, being semisimple, it is stable.

3. Stable Quantum Relativistic Kinematics

Hopefully, the above example will have aroused the interest of the reader enough to follow us as we embark on the search for a stable Lie algebra, encompassing relativistic and quantum effects. Our starting point is the fourteen generator Poincaré-plus-positions algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}) \,, \tag{46}$$

$$[J_{\rho\sigma}, P_\mu] = i(g_{\mu\sigma}P_\rho - g_{\mu\rho}P_\sigma) \,, \tag{47}$$

$$[J_{\rho\sigma}, Z_\mu] = i(g_{\mu\sigma}Z_\rho - g_{\mu\rho}Z_\sigma) \,, \tag{48}$$

augmented by a central generator M, to appear later in the r.h.s. of the Heisenberg commutator. We follow the practice of omitting all zero commutators, the metric used is $g = \mathrm{diag}(1, -1, -1, -1)$ and c, the speed of light, is taken equal to 1. The resulting fifteen generator algebra, describing classical ($\hbar = 0$) relativistic kinematics (plus the extra generator M) we call $\mathcal{G}_{\mathrm{CR}}$ ("Classical Relativity"). The reader might want to identify $J_{\mu\nu}$ with the Lorentz algebra generators and P_μ with the momenta (and, even, Z_μ with the positions) but we will focus initially on the strictly algebraic problem of stability, and only digress on interpretational aspects, which hold some surprises, in Section 4.

The 2-cochain μ_{CR}, corresponding to $\mathcal{G}_{\mathrm{CR}}$, is given by

$$\mu_{\mathrm{CR}} = \frac{1}{2}\,\Pi^{\alpha\rho}\Pi_\rho{}^\beta \otimes J_{\alpha\beta} + \Pi^{\alpha\rho}\Pi_\rho \otimes P_\alpha + \Pi^{\alpha\rho}\Pi_{\dot\rho} \otimes Z_\alpha \,. \tag{49}$$

A straightforward calculation shows that $[\![\mu_{\mathrm{CR}}, \mu_{\mathrm{CR}}]\!] = 0$, confirming that the Jacobi identity is satisfied in $\mathcal{G}_{\mathrm{CR}}$.

3.1. *Calculation of $H^2(\mathcal{G}_{\mathrm{CR}})$*

We computed the second cohomology group $H^2(\mathcal{G}_{\mathrm{CR}})$ with the help of MATHEMATICA.[k] We did this in two independent ways. In the first one, the components of cochains were calculated explicitly, one-by-one, while in the second a symbolic approach was followed, dealing, e.g., with sums of the form $\Pi^\mu\Pi^\nu \otimes J_{\mu\nu}$ without

[k] We found version 5 a spectacular improvement over its predecessors in handling large systems of linear equations.

expanding them further. The first approach has the advantage of generality, as it can deal, practically without further fine-tuning, with any Lie algebra — details of the calculation are given in the appendix (see Appendix A). The second approach is generally faster, at the price of adjustments needed every time a new object (e.g., an invariant tensor) is introduced. In both approaches, the remark made after Eq. (1) shows that we may consider only Lorentz scalars, drastically reducing the workload. We have, nevertheless, implemented this only in our second approach, to keep the first as general as possible. The result of both calculations is

$$H^2(\mathcal{G}_{\rm CR}) = \{[0], [\psi_{\rm H}], [\psi_{\rm PMZ}], [\psi_{\rm ZMP}], [\psi_{\rm PMP}], [\psi_{\rm ZMZ}]\}\,, \tag{50}$$

i.e., there are five non-trivial generators, with representatives given by

$$\psi_{\rm H} = \Pi^\mu \Pi_{\dot\mu} \otimes M\,, \tag{51}$$

$$\psi_{\rm PMZ} = \Pi^\mu \Pi^M \otimes Z_\mu\,, \tag{52}$$

$$\psi_{\rm ZMP} = \Pi^{\dot\mu} \Pi^M \otimes P_\mu\,, \tag{53}$$

$$\psi_{\rm PMP} = \Pi^\mu \Pi^M \otimes P_\mu\,, \tag{54}$$

$$\psi_{\rm ZMZ} = \Pi^{\dot\mu} \Pi^M \otimes Z_\mu\,. \tag{55}$$

As in the example of Section 2.6, we adopt a compact notation where undotted indices in forms refer to P's, dotted ones to Z's, so that, e.g., $\langle \Pi^{\dot\mu}, Z_\nu \rangle = \delta_\nu{}^\mu$. As before, dots make no difference in Kronecker deltas or epsilon tensors. With a slight abuse of notation, Π^M denotes the 1-form that detects the generator M. Our first code mentioned above, running on a 2 GHz, 256 K, Pentium 4 machine, takes slightly over 2 minutes to arrive at Eq. (50).

3.2. *Finite deformations of $\mathcal{G}_{\rm CR}$*

Each of the cocycles in Eqs. (51)–(55) represents a direction of a possible infinitesimal deformation. For example, the first of these, $\psi_{\rm H}$, when added to $\mu_{\rm CR}$, adds the Heisenberg commutator to $\mathcal{G}_{\rm CR}$, while each of the other four renders M non-central. There are two questions that arise now:

(1) Are these infinitesimal deformations integrable?
(2) Are linear combinations of these infinitesimal deformations integrable?

To this end, we compute the commutators among the ψ's and find that the only non-zero ones are those between $\psi_{\rm H}$ and the rest of the ψ's — it will prove convenient in what follows to use the linear combinations $\psi_- = \psi_{\rm ZMZ} - \psi_{\rm PMP}$ and $\psi_+ = \psi_{\rm ZMZ} + \psi_{\rm PMP}$,

$$[\![\psi_{\rm H}, \psi_{\rm PMZ}]\!] = -\Pi^\mu \Pi^\nu \Pi_{\dot\nu} \otimes Z_\mu\,, \tag{56}$$

$$[\![\psi_{\rm H}, \psi_{\rm ZMP}]\!] = \Pi^\nu \Pi^{\dot\mu} \Pi_{\dot\nu} \otimes P_\mu\,, \tag{57}$$

$$[\![\psi_{\rm H}, \psi_-]\!] = \Pi^\mu \Pi^\nu \Pi_{\dot\nu} \otimes P_\mu + \Pi^\nu \Pi^{\dot\mu} \Pi_{\dot\nu} \otimes Z_\mu\,, \tag{58}$$

$$[\![\psi_{\rm H}, \psi_+]\!] = \Pi^\nu \Pi^{\dot\mu} \Pi_{\dot\nu} \otimes Z_\mu - \Pi^\mu \Pi^\nu \Pi_{\dot\nu} \otimes P_\mu - 2\Pi^M \Pi^\nu \Pi_{\dot\nu} \otimes M\,. \tag{59}$$

Regarding the first question above, the fact that the diagonal commutators are all zero implies that $\mu_{\mathrm{CR}} + t\psi_A$, for t finite, gives a deformation of $\mathcal{G}_{\mathrm{CR}}$, where ψ_A is any of the five generators of $H^2(\mathcal{G}_{\mathrm{CR}})$ given above, Eqs. (51)–(55). For the second question, the fact that the commutators among the last four ψ's are all zero implies that any linear combination of these ψ's gives rise to a finite deformation as above. The case of deformations that mix ψ_{H} with the other four generators needs special treatment. We consider an infinitesimal deformation along the 2-cocycle ϕ_1,

$$\phi_1 = q\psi_{\mathrm{H}} + \beta_1\psi_{\mathrm{PMZ}} + \beta_2\psi_{\mathrm{ZMP}} + \beta_-\psi_- + \beta_+\psi_+ \,. \tag{60}$$

One easily checks that $[\![\phi_1, \phi_1]\!] \neq 0$ (the relevant anticommutators are given in Eqs. (56)–(59) above), so that (28) is not trivially satisfied. For the above mentioned anticommutators we find that the first three are trivial,

$$[\![\psi_{\mathrm{H}}, \psi_{\mathrm{PMZ}}]\!] = -D_{\mu_{\mathrm{CR}}}\psi_{\mathrm{PPJ}}\,, \tag{61}$$

$$[\![\psi_{\mathrm{H}}, \psi_{\mathrm{ZMP}}]\!] = D_{\mu_{\mathrm{CR}}}\psi_{\mathrm{ZZJ}}\,, \tag{62}$$

$$[\![\psi_{\mathrm{H}}, \psi_-]\!] = -D_{\mu_{\mathrm{CR}}}\psi_{\mathrm{PZJ}}\,, \tag{63}$$

where

$$\psi_{\mathrm{PPJ}} = \frac{1}{2}\Pi^\mu\Pi^\nu \otimes J_{\mu\nu}\,, \tag{64}$$

$$\psi_{\mathrm{ZZJ}} = \frac{1}{2}\Pi^{\dot\mu}\Pi^{\dot\nu} \otimes J_{\mu\nu}\,, \tag{65}$$

$$\psi_{\mathrm{PZJ}} = \Pi^\mu\Pi^{\dot\nu} \otimes J_{\mu\nu}\,. \tag{66}$$

On the other hand, $[\![\psi_{\mathrm{H}}, \psi_+]\!]$ turns out to be non-trivial. Accordingly, the infinitesimal deformation generated by ϕ_1 might be integrable if, and only if, $\beta_+ = 0$. In that case, Eq. (28) is satisfied with

$$\phi_2 = q\beta_1\psi_{\mathrm{PPJ}} - q\beta_2\psi_{\mathrm{ZZJ}} + q\beta_-\psi_{\mathrm{PZJ}}\,. \tag{67}$$

With an eye on (29), we compute $[\![\phi_1, \phi_2]\!]$ and find that it vanishes, so that $\phi_3 = 0$ (see Eq. (29)). Also, $[\![\phi_2, \phi_2]\!] = 0$, implying that ϕ_4, and all higher order ϕ_n's vanish, and the finite deformation truncates at second order,

$$\begin{aligned}\mu_{\mathrm{CR}} + \phi_t &= \mu_{\mathrm{CR}} + \phi_1 t + \phi_2 t^2 \\ &= \mu_{\mathrm{CR}} + (q\psi_{\mathrm{H}} + \beta_1\psi_{\mathrm{PMZ}} + \beta_2\psi_{\mathrm{ZMP}} + \beta_-\psi_-)t \\ &\quad + q(\beta_1\psi_{\mathrm{PPJ}} - \beta_2\psi_{\mathrm{ZZJ}} + \beta_-\psi_{\mathrm{PZJ}})t^2\,.\end{aligned} \tag{68}$$

Without loss of generality, we may put $t = 1$ and write the result as

$$\begin{aligned}\mu_{\mathrm{CR}} + \phi_{t=1} &= \mu_{\mathrm{CR}} + q\psi_{\mathrm{H}} + \beta_1(\psi_{\mathrm{PMZ}} + q\psi_{\mathrm{PPJ}}) \\ &\quad + \beta_2(\psi_{\mathrm{ZMP}} - q\psi_{\mathrm{ZZJ}}) + \beta_-(\psi_- + q\psi_{\mathrm{PZJ}})\,,\end{aligned} \tag{69}$$

a form that will prove useful shortly.

3.3. *Heisenberg's route*: *The algebra* $\mathcal{G}_{\rm PH}(q)$

We want to explore here what happens if one follows, along with Heisenberg, the historical route and only deforms $\mathcal{G}_{\rm CR}$ along $\psi_{\rm H}$. We consider, accordingly, the stability of the algebra $\mathcal{G}_{\rm PH}(q)$ ("Poincaré plus Heisenberg"), with corresponding 2-cochain $\mu_{\rm PH}(q)$ given by

$$\mu_{\rm PH}(q) = \mu_{\rm CR} + q\psi_{\rm H} \tag{70}$$

(we assume henceforth that $q \neq 0$). The commutators defining it are given by Eqs. (46)–(48), plus the Heisenberg commutator — for the sake of locality we collect them all here,

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho})\,, \tag{46$'$}$$

$$[J_{\rho\sigma}, P_{\mu}] = i(g_{\mu\sigma}P_{\rho} - g_{\mu\rho}P_{\sigma})\,, \tag{47$'$}$$

$$[J_{\rho\sigma}, Z_{\mu}] = i(g_{\mu\sigma}Z_{\rho} - g_{\mu\rho}Z_{\sigma})\,, \tag{48$'$}$$

$$[P_{\mu}, Z_{\nu}] = iqg_{\mu\nu}M\,. \tag{71}$$

We will have more to say about (71) in Section 4 — for the moment, we ask the reader to accept it as a reasonable (i.e., covariant and of Lie-type) form of the familiar Heisenberg relation. As in the previous case, of $\mathcal{G}_{\rm CR}$, we first tackle the purely algebraic problem of stability, and leave questions of physical interpretation for Section 4. Meanwhile, the temptation should be resisted to consider $\mathcal{G}_{\rm PH}$ as the algebra of "quantum relativistic kinematics" — it is argued later on that it is not.

We find that $H^2(\mathcal{G}_{\rm PH}(q))$ is non-trivial,

$$H^2(\mathcal{G}_{\rm PH}(q)) = \{[0], [\zeta_1], [\zeta_2], [\zeta_3]\}\,, \tag{72}$$

where

$$\zeta_1 = \psi_{\rm PMZ} + q\psi_{\rm PPJ}\,, \tag{73}$$

$$\zeta_2 = -\psi_{\rm ZMP} + q\psi_{\rm ZZJ}\,, \tag{74}$$

$$\zeta_3 = \psi_{-} + q\psi_{\rm PZJ}\,. \tag{75}$$

$\psi_{\rm H}$ itself is still a cocycle, albeit a trivial one now (for all non-zero q). This means that, starting at $\mathcal{G}_{\rm PH}(q)$, and moving along $\psi_{\rm H}$, one arrives at isomorphic algebras. But moving along $\psi_{\rm H}$ amounts to changing the value of q, without changing its sign. We conclude that the algebras $\mathcal{G}_{\rm PH}(q)$, for all non-zero values of q, of the same sign, are isomorphic. On the other hand, one can easily change the sign of q by a redefinition of the generators, e.g., by rescaling all Z's by some negative number, or exchanging P and Z (notice that in both examples, the corresponding matrix that effects the redefinition has positive determinant, i.e., it lies in the connected component of $\mathrm{GL}(15,\mathbb{R})$). The upshot of all this is that all $\mathcal{G}_{\rm PH}(q)$, for non-zero q, are isomorphic.

3.4. Finite deformations of $\mathcal{G}_{\mathrm{PH}}(q)$

Reasoning as in Section 3.2, we compute the commutators among the ζ's, and find that they all vanish. Accordingly, every linear combination of the above cocycles, added to μ_{PH}, provides a finite deformation of $\mathcal{G}_{\mathrm{PH}}$. For a generic combination $\zeta(\boldsymbol{\alpha})$,

$$\zeta(\boldsymbol{\alpha}) = \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3 \,, \tag{76}$$

the 2-cochain $\mu(q,\boldsymbol{\alpha}) = \mu_{\mathrm{PH}}(q) + \zeta(\boldsymbol{\alpha})$, is clearly identical to the one found before, Eq. (69), with the identifications $(\beta_1, \beta_2, \beta_-) \mapsto (\alpha_1, -\alpha_2, \alpha_3)$. We have arrived then at the same result, either deforming $\mathcal{G}_{\mathrm{CR}}$ along a direction that truncates to second order in the deformation parameter, or by performing two successive deformations (with intermediate stop at $\mathcal{G}_{\mathrm{PH}}(q)$), each truncating to first order. The corresponding deformed algebra is given by the commutators of $\mathcal{G}_{\mathrm{CR}}$, Eqs. (46)–(48) (notice that the Heisenberg commutator is not included), plus the following

$$[P_\mu, Z_\nu] = iqg_{\mu\nu}M + iq\alpha_3 J_{\mu\nu} \,, \tag{77}$$

$$[P_\mu, P_\nu] = iq\alpha_1 J_{\mu\nu} \,, \tag{78}$$

$$[Z_\mu, Z_\nu] = iq\alpha_2 J_{\mu\nu} \,, \tag{79}$$

$$[P_\mu, M] = -i\alpha_3 P_\mu + i\alpha_1 Z_\mu \,, \tag{80}$$

$$[Z_\mu, M] = -i\alpha_2 P_\mu + i\alpha_3 Z_\mu \,. \tag{81}$$

We denote the resulting algebra by $\mathcal{G}_{\mathrm{PH}}(q,\boldsymbol{\alpha})$. We see that, for a generic deformation, the P's cease to commute among themselves, and so do the Z's, M is no longer central, while the Heisenberg commutator receives an additional term, proportional to $J_{\mu\nu}$.

3.5. The instability cone

Relevant questions that emerge now are:

(1) Are the above deformations $\mathcal{G}_{\mathrm{PH}}(q,\boldsymbol{\alpha})$, for various values of $\boldsymbol{\alpha}$, (finally) stable?
(2) Are there deformations that are isomorphic among themselves?

To answer the first question, we compute, as always, the second cohomology group and find

$$H^2(\mathcal{G}_{\mathrm{PH}}(q,\boldsymbol{\alpha})) = \begin{cases} \{[0]\} & \text{if } \alpha_3^2 \neq \alpha_1\alpha_2 \\ \{[0],[\chi]\} & \text{if } \alpha_3^2 = \alpha_1\alpha_2 \end{cases} , \tag{82}$$

where $\chi = \zeta_1 + \zeta_2$ satisfies $[\![\chi,\chi]\!] = 0$. $\mathcal{G}_{\mathrm{PH}}(q,\boldsymbol{\alpha})$ is, accordingly, stable everywhere outside the instability surface $\alpha_3^2 = \alpha_1\alpha_2$ in α-space. The latter represents a double cone with the apex at the origin and its axis along the first diagonal in the α_1–α_2 plane, parallel to χ (see Fig. 2). We will refer to the various regions of α-space with their relativistic nicknames ("future", "past", etc.), with the future including the positive α_1–α_2 quadrant. Notice that, off the cone, χ is a trivial cocycle, $\chi = \nabla\xi$, and

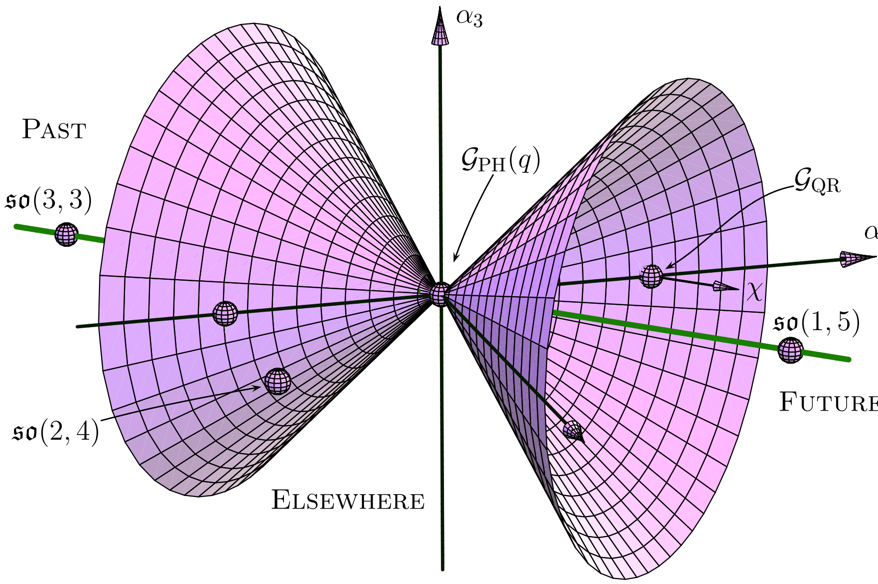

Figure 2. The $(\alpha_1, \alpha_2, \alpha_3)$ deformation space of $\mathcal{G}_{\rm PH}(q)$, with a representative of each of the six equivalence classes drawn (these are, in relativistic parlance, the future and past light-cones, the apex, the future, the past and the elsewhere). The two cones and the apex at the origin correspond to unstable algebras — the rest of the space to stable ones. For all classes, a representative exists with $\alpha_3 = 0$ (the little spheres denote such representatives). With the identification of Z_μ with the moment operator (see Section 4), the origin corresponds to $\mathcal{G}_{\rm PH}(q)$, while $\mathcal{G}_{\rm QR}$, the Lie algebra of standard quantum relativistic kinematics, lies on the future cone, at $\boldsymbol{\alpha} = (0, q, 0)$. Stabilizing deformations of $\mathcal{G}_{\rm QR}$, generated, e.g., by χ, may lead either towards the future (isomorphic to $\mathfrak{so}(1,5)$) or towards the elsewhere (isomorphic to $\mathfrak{so}(2,4)$). Both choices introduce non-commutativity of the P's, differing in the sign of the associated curvature.

ξ has a pole on the cone. Regarding the second question above, from the fact that each algebra outside the light cone is isomorphic to all algebras in its neighborhood, we conclude that all algebras in, say, the future, are isomorphic among themselves (similarly for the past and the elsewhere). A slight refinement of the argument, using the fact that tangent vectors to the cone are trivial cocycles, leads to the conclusion that all algebras in, say, the future cone, are isomorphic among themselves (similarly for the past cone), with the apex, i.e., $\mathcal{G}_{\rm PH}(q, \mathbf{0}) \equiv \mathcal{G}_{\rm PH}(q)$, in a class by itself. In conclusion, *there are six equivalence classes of algebras, given by the various regions the α-space is divided into by the double light cone.*

A glance at Fig. 2 helps visualize several aspects of the stability analysis that were mentioned earlier (see Section 2.4). For example, starting at some (unstable) point of, say, the future cone, it is clear that infinitesimal deformations along the cocycle χ given above lead either to the future or to the elsewhere, both being non-isomorphic to the original algebra but also between themselves. Since $[\![\chi, \chi]\!] = 0$, one may also consider finite deformations,

$$\mu_t = \mu_{\rm PH}(q, \alpha_1, \alpha_2, \sqrt{\alpha_1 \alpha_2}) + t\chi \,. \tag{83}$$

In this case, it is clear that there exists a negative value of t ($t_0 = -\alpha_1 - \alpha_2$) such that the resulting algebra $\mu_{t_0} = \mu_{\mathrm{PH}}(q, -\alpha_2, -\alpha_1, \sqrt{\alpha_1\alpha_2})$ lies on the past light cone and is, therefore, non-isomorphic to μ_t, for generic t. Finally, had we chosen instead a non-trivial cocycle orthogonal to the axis of the cone, rather than parallel to it,[1] there would exist a finite deformation along it isomorphic to the original algebra, given by the "antipodal" point on the future cone, $(\alpha_2, \alpha_1, -\sqrt{\alpha_1\alpha_2})$.

3.6. *Gauging away the* $\boldsymbol{\alpha}$*'s*

The above conclusion makes it evident that, for each of the above classes, a representative exists with $\alpha_3 = 0$. The deformation space then, from the algebraic point of view, is essentially the α_1–α_2 plane. To find explicitly a linear redefinition of the generators that moves an arbitrary point in α-space to the α_1–α_2 plane, we notice that the transformation $(P, Z) \mapsto (P', Z')$, given by

$$P'_\mu = aP_\mu + bZ_\mu\,, \qquad Z'_\mu = cP_\mu + dZ_\mu\,, \qquad ad - bc = 1\,, \tag{84}$$

leaves the value of q in the Heisenberg commutator invariant, while, in α-space, it induces the transformation $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}' = R\boldsymbol{\alpha}$, where

$$R = \begin{pmatrix} a^2 & b^2 & 2ab \\ c^2 & d^2 & 2cd \\ ac & bd & ad + bc \end{pmatrix}. \tag{85}$$

It can be checked that, in the particular case where the P-Z transformation is a rotation by an angle θ, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, the matrix R that results from (85) describes a rotation in α-space, around the axis of the cone, by an angle of 2θ, counterclockwise as seen from the future. Such a rotation clearly leaves all isomorphism classes invariant and can be chosen so as to move any particular point to the α_1–α_2 plane. Similarly, for algebras in the elsewhere, isomorphic algebras exist with either α_1 or α_2 equal to zero — this is not true for algebras in the past or the future (the orbits of the latter under the above rotation do not intersect the α_1–α_3 or α_2–α_3 planes). Having said that, due consideration should be given to the fact that physicists can be single-minded in what regards their preferred set of generators, e.g., working with arbitrary linear combinations of momenta and positions could be frowned on. If such preferences are given priority, one might be forced to work with a version of the deformed algebra with $\alpha_3 \neq 0$.

3.7. *Isomorphisms*

This brings our stability analysis to a conclusion. The algebra $\mathcal{G}_{\mathrm{PH}}(q, \boldsymbol{\alpha})$ we have arrived at is given by Eqs. (46)–(48) and (77)–(81), with corresponding 2-cochain

$$\mu_{\mathrm{PH}}(\boldsymbol{\alpha}) = \mu_{\mathrm{CR}} + q\psi_{\mathrm{H}} + \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3\,. \tag{86}$$

[1] This can achieved by adding an appropriate trivial cocycle to χ.

The novelties are that the P's don't commute, the Z's do not commute, M is no longer central, and an extra term appears in the Heisenberg commutator. The nature of the three deformation parameters α_i is discussed in Section 4.5, after the physical identification of the generators has been carried out.

As it has been pointed out in Refs. 19 and 27, the above algebra, off the instability cone, is isomorphic to some $\mathfrak{so}(m, 6-m)$, where m depends on the signs of α_1, α_2 (taking $\alpha_3 = 0$). The isomorphism, given in Ref. 27, is as follows: denote the generators of $\mathfrak{so}(m, 6-m)$ by $\{J_{\mu\nu}, J_{\mu 4}, J_{\mu 5}, J_{45}\}$ — their commutation relations are analogous to those of the Lorentz group, Eq. (46), with a metric $\bar{g}$ that is taken diagonal, with entries ± 1, and coinciding with g in the Lorentz sector. Assuming the identifications

$$P_\mu = \sigma J_{\mu 4}\,, \quad Z_\mu = \tau J_{\mu 5}\,, \quad M = \rho J_{45}\,, \tag{87}$$

one finds the commutators

$$[P_\mu, Z_\nu] = -i\,\frac{\sigma\tau}{\rho}\,\bar{g}_{\mu\nu} M\,, \quad [P_\mu, M] = i\,\frac{\rho\sigma}{\tau}\,\bar{g}_{44} Z_\mu\,, \quad [Z_\mu, M] = -i\,\frac{\rho\tau}{\sigma}\,\bar{g}_{55} P_\mu\,. \tag{88}$$

Comparing the first of these with the Heisenberg commutator gives $\rho = -q\sigma\tau$. Substituting this in the other two, and comparing with (80), (81), respectively, shows that

$$\bar{g} = \mathrm{diag}(1, -1, -1, -1, \epsilon_4, \epsilon_5)\,, \quad \epsilon_4 \equiv -\mathrm{sgn}(q\alpha_1)\,, \quad \epsilon_5 \equiv -\mathrm{sgn}(q\alpha_2)\,, \tag{89}$$

i.e., assuming $q > 0$,

$$\mathcal{G}_{\mathrm{PH}}(q, \alpha_1, \alpha_2, \alpha_3 = 0) \cong \begin{cases} \mathfrak{so}(1,5) & \text{if } \alpha_1 > 0,\ \alpha_2 > 0 \\ \mathfrak{so}(2,4) & \text{if } \alpha_1\alpha_2 < 0 \\ \mathfrak{so}(3,3) & \text{if } \alpha_1 < 0,\ \alpha_2 < 0 \end{cases}\,. \tag{90}$$

(see Fig. 2). On the light-cone, the above semisimple (and, hence, stable) algebras go over to the corresponding semidirect product.[19]

3.8. *Relation with other algebras*

As mentioned in the Introduction, several non-commutative spacetime Lie algebras have been proposed over the years, but, as a rule, they fail to provide a complete set of generators. We provide below an account of these earlier attempts, emphasizing from the outset that our list of references does not pretend to be complete.

The first, to our knowledge, publication regarding a non-commutative, Lorentz covariant spacetime is due to Snyder, dating from 1947. Apparently, as J. Wess has documented and publicized, the idea can be traced back to Heisenberg, who, in a letter to Peierls, suggested that the ultraviolet infinities of quantum field theory could be tamed by assuming non-commuting spacetime coordinates. Peierls soon found an altogether different application in the calculation of the lowest Landau level of a system of electrons in a magnetic field with impurities, using non-commuting coordinates in the potential-like function describing them. He also passed on the idea to Pauli, who described it to Oppenheimer, who shared it with Snyder, who published Ref. 36 (our source is Ref. 17). Snyder's position operators fail to commute among themselves, exactly as in (79). His momenta, however, commute, and the position-momenta relations contain non-linear terms. An early attempt at formulating electrodynamics in this non-commutative spacetime followed shortly after.[35] Later in that year, Yang,[39] pointed out that by introducing what we have called M, one can render the algebra linear. Additionally, he proposed non-commuting momenta, exactly as in (78) *and* the accompanying P-M relations, Eq. (80) (with $\alpha_3 = 0$). The $\mathfrak{so}$ isomorphism was also given, although with a particular choice for the signs ϵ_4, ϵ_5 (both equal to -1). In fact, getting to some known, preferably semisimple, algebra (like $\mathfrak{so}$), seems to have been his guiding principle in completing the set of commutators.

Several years later, Khruschev and Leznov[19] provided a further deformation, essentially the one given by our ζ_3, i.e., by the terms proportional to α_3 in (77)–(81). Although their article cited above appeared in 2002, it quotes this result (or, at least, significant parts of it) from an earlier work of theirs, dating from 1973, which we have not had access to. Their approach is via a straightforward solution of the Jacobi identities, and does not include the stability point of view. Apart from the deformation itself, they also provide information on the Casimirs of the deformed algebras, and mention the $\alpha_1\alpha_2 - \alpha_3^2 \neq 0$ relation, with $\alpha_1\alpha_2 \neq 0$, as a semisimplicity criterion for the deformed algebra. The $\mathfrak{so}$ identifications are given (they actually use $\mathfrak{o}(n, 6-n)$) but the possibility of gauging away α_3 by a redefinition of the generators is not pointed out. Some steps towards constructing field theories on these quantized spaces were taken in Ref. 23.

The work of Vilela Mendes,[27] which motivated ours, appeared in 1994. There, for the first time, the stability criterion for the non-commutative spacetime algebra is invoked and its relevance, more generally, for the algebraic structures employed in physical theories is convincingly advocated. The approach taken in determining the stable form of the algebra is a minimalistic one: the $\mathfrak{so}$ algebras are proposed *ab initio*, being obviously deformations, and their semisimplicity is invoked to guarantee their stability. Economical as it may be this approach, it leaves nevertheless pending the question of uniqueness, prompting us to undertake the present systematic search. The instability double cone is not mentioned in the above work. Also, although Snyder's work was known to the author, it seems he did not come across Yang's contribution. Various applications have been considered by Vilela Mendes and co-workers in Refs. 5, 26 and 28.

In recent years, as mentioned in the Introduction, several "Doubly Special Relativity" algebras have been proposed. They all ignore the (initially) central generator M. In Ref. 20 it was shown that all commutation relations of the deformed algebras, except the P-Z ones, can be brought into a Lie form by appropriate non-linear redefinitions of the generators. The Lie form found coincides with that provided by the α_2 deformation above. Furthermore, it was pointed out in Ref. 9, that by taking M into account, the "Triply Special Relativity" of Ref. 21 is linearized, and the resulting, Lie-type, deformation is the one provided by α_1 above (this was essentially a repeat of Yang's observation on Snyder's proposal, applied to the momentum sector). Thus, it seems that when M is taken into account, non-linear redefinitions bring the "Multi-Special Relativity" algebras into one of the forms found above.[m] These observations suggest that, before leaving the tried and tested Lie algebra framework, Lie deformations introducing new invariant scales, like the ones proposed earlier, should be studied carefully, and the need for non-linearity should be critically examined.

Finally, the following obvious fact should be emphasized: isomorphic, or even identical, algebras may correspond to radically different physics if the generators that enter in them are interpreted in different ways. In this respect, all of the above mentioned works coincide in the physical identification of the generators, in particular, in the fact that the Z_μ's should be interpreted as position operators. As we explain in the section that follows, our view differs.

4. Some Physical Considerations

We deal, finally, with a number of interpretational issues. We would like to warn the reader that the material in this section is still in its formative stage, and several aspects of what follows are still under investigation. Nevertheless, we feel it is worthwhile pointing out alternative possibilities in the physical identification of the generators we have been studying. The content here is mostly qualitative and the tone, accordingly, informal. We keep complexity to a minimum by treating the case of a massive, spinless particle only — a more complete analysis will have to wait, like so many other things, a future work.

4.1. *The coproduct of Lie algebra generators*

We wish to discuss the physical meaning of the coproduct of Lie algebra generators. We will use the Poincaré algebra as an example, but the discussion applies to general Lie algebras.

Consider applying a translation $T_{\mathbf{a}}$ to a particle, located at $\mathbf{x}$. As a result, the particle shifts to $\mathbf{x}+\mathbf{a}$. Imagine now that, under closer inspection, the particle is seen to be a bound state of two other particles. To translate by $\mathbf{a}$ what is now known to be a two-particle system, one applies the translation $T_{\mathbf{a}}$ to each of the constituent particles in the system. The n-particle case, $n>2$, is handled by further subdivision of either of the two particles above. Similar considerations hold for rotations or boosts. This observation is formalized in the following manner. The state of the system under study is represented by a state vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$. To a possible transformation of the system, e.g., a rotation $R_{\alpha\beta\gamma}$ parametrized by Euler's angles, one associates an operator $\mathcal{D}(R_{\alpha\beta\gamma})$, acting on $\mathcal{H}$. When the system

[m] This statement is meant as an observation of a pattern, not as a theorem — we have certainly not checked each and every non-linear algebra proposed.

is revealed to consist of, say, particles 1 and 2, the state space becomes $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i$ is the state space of particle i. The observation made above then implies that the operator representing $R_{\alpha\beta\gamma}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is simply $\mathcal{D}_1(R_{\alpha\beta\gamma}) \otimes \mathcal{D}_2(R_{\alpha\beta\gamma})$, where $\mathcal{D}_i$ is the representation of rotations in $\mathcal{H}_i$. This is true for all representations $\mathcal{D}_i$ — we may accordingly conclude that the abstract rotation operator $R_{\alpha\beta\gamma}$ acts on tensor products as $R_{\alpha\beta\gamma} \otimes R_{\alpha\beta\gamma}$ and call this latter operator the *coproduct* $\Delta(R_{\alpha\beta\gamma})$ of $R_{\alpha\beta\gamma}$. Particular cases then are handled by taking the appropriate representation of this universal formula, e.g., $\mathcal{D}_1 \otimes \mathcal{D}_2$ above. The fact that rotations should compose in the same way, whether applied to a simple or to a composite system, is expressed algebraically by the requirement that[n] $\Delta(R_1 R_2) = \Delta(R_1)\Delta(R_2)$. Our formalism respects this for, if $R_1 R_2 = R_3$, then

$$\begin{aligned} \Delta(R_1 R_2) &= \Delta(R_3) \\ &= R_3 \otimes R_3 \\ &= R_1 R_2 \otimes R_1 R_2 \\ &= (R_1 \otimes R_1)(R_2 \otimes R_2) \\ &= \Delta(R_1)\Delta(R_2)\,, \end{aligned} \tag{91}$$

the product law in the tensor product being $(A \otimes B)(C \otimes D) = AC \otimes BD$. We summarize: for all transformations T in the Poincaré group

- the coproduct $\Delta(T)$ is *grouplike*,

$$\Delta(T) = T \otimes T \tag{92}$$

- Δ is an algebra homomorphism,

$$\Delta(T_1 T_2) = \Delta(T_1)\Delta(T_2)\,. \tag{93}$$

Now write $T = e^A$, with A in the Poincaré algebra $\mathcal{G}_{\mathrm{P}}$ and *define* Δ to be linear in the entire $U(\mathcal{G}_{\mathrm{P}})$, the universal enveloping algebra[o] of $\mathcal{G}_{\mathrm{P}}$ — a simple calculation then shows that $\Delta(A) = A \otimes 1 + 1 \otimes A$ (this is a logarithm turning a product into a sum, as usual). We conclude that

- The generators of grouplike transformations are *primitive*,

$$\Delta(A) = A \otimes 1 + 1 \otimes A\,, \tag{94}$$

with $J_{\mathrm{tot}} = J_1 + J_2$ as the archetypical example from quantum mechanics. In other words, *the physical quantities corresponding to generators of grouplike transformations are additive under system composition* (or *extensive*, in thermodynamics parlance). All Lie algebra generators are of this nature. From a geometric point

[n] Another way of writing this is $[\Delta(T_A), \Delta(T_B)] = f_{AB}{}^{C}\Delta(T_C)$, i.e., the generators $\Delta(T_A)$ in $\mathcal{G} \otimes \mathcal{G}$ satisfy the same commutation relations as the T_A's.

[o] More precisely, a certain topological completion of $U(\mathcal{G}_{\mathrm{P}})$.

of view, the definition of pointwise multiplication of functions on the group manifold, $(fh)(g) = f(g)h(g)$, is what fixes the coproduct of the point (transformation) g to be grouplike. At the infinitesimal level, this becomes the Leibniz rule satisfied by the generators (this is another way of interpreting (94)), consistent with their representation as first-order differential operators on the group manifold. The above considerations prompt us to only allow primitive operators as Lie algebra generators.

In Ref. 27, it is argued that the one-dimensional Heisenberg commutator, $[p, x] = -i$, can be interpreted as defining a stable Lie algebra. The justification for this claim is made through the observation that one could equally well choose a function of x as a coordinate, in particular, $y = e^{ix}$. Then, the Heisenberg relation takes the form $[p, y] = y$, which indeed defines a stable two-dimensional algebra. Apart from the unsuitability of e^{ix} as coordinate over the entire x-axis, it should be clear from our earlier discussion that we cannot agree with this argument, since y is no more primitive than x.

4.2. *The Lie form of the Heisenberg algebra*

As mentioned after our first reference to the Heisenberg commutator, Eq. (71), there are a number of remarks that we would like to make regarding its proposed form. One usually first encounters the Heisenberg commutator in the form

$$[P_i, X^j] = -iq\delta_i{}^j\,, \tag{95}$$

which is unsatisfactory for (at least) two reasons. The first has to do with Lorentz covariance — the obvious remedy is to consider instead the form

$$[P_\mu, X_\nu] = iqg_{\mu\nu}\,, \tag{96}$$

leaving for a future brainstorm the elucidation of its physical implications (notice that time is promoted to an operator). The second reason is of a technical nature: dealing with a Lie algebra, the r.h.s. of (96) ought to be linear in the generators. The usual solution followed in the literature is to introduce a new, central generator M, with

$$[P_\mu, X_\nu] = iqg_{\mu\nu}M\,. \tag{97}$$

The resulting three-generator Lie algebra is referred to as the Heisenberg algebra — the physical interpretation of M is generally left obscure. It might at first seem that there is little to be gained from writing out M explicitly, since it commutes with everything, but when deformations of the algebra are considered, it will be essential to do so since, as a result of the deformation, M might cease to be central (for an example of the type of problems that may arise by suppressing M, see Ref. 9).

Is (97), at last, an acceptable form of the Heisenberg algebra? That the answer should still be negative follows easily from our remarks about the primitiveness of Lie algebra generators. First, if M in the r.h.s. of (97) were primitive (and hence extensive), the effective Planck's constant for a composite system would be the sum of those for its constituent parts, providing for several concrete examples of fuzzy spheres (e.g., the earth, with $q_{\mathrm{Earth}} \cong 10^{14}$ Kgr m^2/sec). Second, X_μ is not primitive. There are various ways to see this. To begin with, it is rather obvious that position is not an extensive quantity: if two particles are glued together at x_μ, their composite system is also located at x_μ, not at $2x_\mu$. Another way is to look at the corresponding finite transformation. X_μ may be considered, up to a sign, the generator of translations in momentum space. But the apparent symmetry (via duality) between momenta and positions should be treated with care. In particular, although translations in spacetime are grouplike, those in momentum space are not. If a particle of 4-momentum p is translated, in momentum space, by k, it ends up with 4-momentum $p+k$. If now it is discovered that it is actually made up of two other particles and each of them is translated by k, then the composite particle would be translated by $2k$. This latter example reveals something about the nature of the grouplike operator that should replace e^X, the logarithm of which would be acceptable as a Lie algebra generator. Roughly speaking, it should somehow detect the mass of the particle and translate in momentum space by a quantity proportional to it. Notice that, despite the elementary nature of the considerations in this section, there seems to exist a consensus in the literature that X_μ *is* primitive.[p]

4.3. *The coproduct of the position operator*

So, if X_μ is not primitive, what is its coproduct $\Delta(X_\mu)$? The answer is that, in general, $\Delta(X_\mu)$ does not exist. To see why this is so, let us first specify what exactly is it that we want the position operator to do for us. For a single localized particle it is clear that X_μ should return its position, but what should X_μ (via its coproduct $\Delta(X_\mu)$) do on a two-particle system? Clearly, if the two particles are glued together and the composite system is localized, we should get the same answer whether we operate with X_μ on the composite particle or with $\Delta(X_\mu)$ on the two-particle system. When the two particles are far apart and/or have different velocities, the natural requirement would be that $\Delta(X_i)$ (i.e., the spatial part of $\Delta(X_\mu)$) should return the position of their center-of-momentum (or center-of-inertia), i.e., the relativistic refinement of the newtonian center-of-mass concept, which is the natural "effective position" of a relativistic composite system.[q] The problem is that the center-of-momentum 3-vector is not the spatial part of any 4-vector, in other words, the "effective position" of a composite relativistic system does not behave as a 4-vector. As a result, different observers locate the center-of-momentum of a system at different points. This, in turn, implies that $\Delta(X_\mu)$ does not satisfy the same commutation relations with $\Delta(J_{\rho\sigma})$ as X_μ does with $J_{\rho\sigma}$

[p]The references assuming so are too many to list here explicitly — Ref. 20 may nevertheless be singled out for actually *deriving* this result (see their Eq. (26) and the erroneous argument preceding it).

[q]See, for example, the discussion in Ref. 32, p. 84 and Ref. 22, p. 42.

(see footnote n), in other words, Δ fails to be a homomorphism of the algebra, which proves our assertion.

The above conclusion might well be correct, but, certainly, there are composite systems the "effective position" of which, for all practical purposes, behaves like a 4-vector (e.g., an α-particle). This implies that although, strictly speaking, $\Delta(X_\mu)$ does not exist, one might nevertheless define an approximate coproduct that works provided it is applied on a restricted class of systems — intuitively, systems that can fool the observer into thinking they are a single, localized particle. To make this statement precise, we note that the center-of-momentum spatial coordinates of a (non-interacting) two-particle system are given by

$$\mathbf{R} = \frac{E_1 \mathbf{r}_1 + E_2 \mathbf{r}_2}{E} \,, \tag{98}$$

where $E \equiv E_1 + E_2$ is the total energy of the system (this formula makes it clear that $\mathbf{R}$ is not the spatial part of any 4-vector). Assume now[r] that the system under study is such that in its center-of-momentum frame all energies E_i are nearly equal to the corresponding rest masses, $E_i \cong m_i$ — we will call such a system *psychron*, from the greek ψυχρόν for "cold." Then, in the above frame, (98) reduces to the Newtonian formula for the center-of-mass. Moreover, when boosting to an arbitrary frame, all energies in the r.h.s. of (98) transform by the same γ-factor, which cancels, so that the l.h.s. transforms as a spatial vector. We conclude that, for psychron 2-particle systems, the relation

$$m_{12} x_{12}^\mu = m_1 x_1^\mu + m_2 x_2^\mu \,, \tag{99}$$

where $m_{12} \equiv m_1 + m_2$, defines the effective position x_{12} of the system as a 4-vector. Denoting by M the mass operator, $M^2 = P^\mu P_\mu$, and brushing aside ordering ambiguities, we conclude from (99) that the *moment operator* $Z_\mu \equiv X_\mu M$ *is* primitive, when applied to psychron systems (M is also primitive on such systems). We note furthermore that Z_μ is of exactly the form anticipated by the argument at the end of Section 4.2. In terms of Z, the covariant version of the Heisenberg relation, Eq. (96), takes the familiar form used earlier,

$$[P_\mu, Z_\nu] = iq g_{\mu\nu} M \,, \tag{71$'$}$$

albeit with a new interpretation. Notice that, with M positive, and our convention for the metric, the standard quantum mechanical relations correspond to $q = \hbar$.

4.4. *The algebra of standard quantum relativistic kinematics*

We investigate the repercussions of the above interpretation of Z_μ, M, in identifying the algebra of standard quantum relativistic kinematics. In the latter, the momenta commute and so do the positions, while their cross-relations are given

[r]Our simplifying assumptions of non-zero mass and zero spin start taking effect from this point on.

by the Heisenberg commutator, Eq. (96). But then the Z's, in terms of which the algebra should be expressed, do not commute,

$$[Z_\mu, Z_\nu] = iq(X_\mu P_\nu - X_\nu P_\mu)\,, \tag{100}$$

and neither do the Z's with M,

$$[Z_\mu, M] = -iqP_\mu\,, \tag{101}$$

where $[X_\mu, f(P)] = -iq\partial f(P)/\partial P^\mu$ was used.[s] Notice that the Z-Z non-commutativity is a purely quantum ($q \neq 0$) phenomenon and has no connection to spacetime non-commutativity. We recognize the r.h.s. of (100) as (a multiple of) the covariant form of the orbital angular momentum generator, $L_{\mu\nu} = q^{-1}(X_\mu P_\nu - X_\nu P_\mu)$. For a massive, spinless particle then, we have

$$[Z_\mu, Z_\nu] = iq^2 J_{\mu\nu}\,. \tag{102}$$

A look at (79), (81), shows that the above relations, Eqs. (101) and (102), are of exactly the form furnished by the α_2 deformation, with $\alpha_2 = q$. We conclude that, for a massive, spinless particle, the algebra $\mathcal{G}_{\mathrm{QR}}$ of standard quantum relativistic kinematics is given by

$$\mathcal{G}_{\mathrm{QR}} = \mathcal{G}_{\mathrm{PH}}(q, 0, q, 0)\,, \tag{103}$$

i.e., in α-space, it lies on the surface of the future cone, along the α_2-axis, at $\alpha_2 = q$ (see Fig. 2). It is worth noting that, with the above interpretation of the Z's, the Heisenberg algebra by itself does not close, the Z-Z commutator generating the Lorentz group.

4.5. *The nature of the deformations*

The deformation corresponding to ζ_1 introduces non-commutativity among the momenta and renders M non-central. Its origins lie in the instability of the Poincaré algebra, which stabilizes to the simple De Sitter algebras $\mathfrak{so}(1,4)$ or $\mathfrak{so}(2,3)$. The corresponding parameter, α_1, has dimensions $[L]^{-1}[M]$, so that $R \equiv \sqrt{\hbar/\alpha_1}$ is a length, the radius of curvature of the manifold on which the various $\mathfrak{so}$ algebras of Section 3.7 act. It has been suggested in Ref. 27 that, as long as one is interested in the kinematics in the tangent space to the manifold, rather than the group of motions of the manifold itself, one may take the $R \to \infty$ limit, i.e., one may essentially disregard the above deformation. On the other hand, in Ref. 21, the suggestion has been made that R^2 may set the scale for the cosmological constant Λ. In any case, this deformation is a familiar and thoroughly studied one.

[s] Strictly speaking, this relation holds for functions $f(P)$ that can be expanded in power series in P — nevertheless, the commutation relations (100), (101), are consistent with $M^2 = P^\mu P_\mu$ and we do not require anything more.

When the Z_μ are identified with the position operators, the deformation generated by ζ_2 turns on spacetime non-commutativity. α_2 in that case has dimensions $[L][M]^{-1}$, so that $\ell \equiv \sqrt{\hbar\alpha_2}$ is a length, the inverse of which has, in the past, been conjectured to set the scale for the masses of the elementary particles. However, if that were the case, the effects of the deformed commutators would by now have been measured, so this proposal had to be abandoned. A more recent tendency is to regard ℓ as the Planck length, and attribute the non-commutativity to quantum gravity effects (see, e.g., Ref. 21 and references therein). Whatever the interpretation of the new length scale may be, the above identification of the Z's seems to us to suffer from a somewhat incredulous prediction: the extent to which the coordinates of a particle do not commute, i.e., the local "fuzziness" in spacetime due to, e.g., quantum gravity effects, depends, in general, on the position of the origin (since $J_{\mu\nu}$, the particle's angular momentum, does). In particular, the coordinates of a particle at the origin commute. We think it improbable that such a state of affairs can be successfully incorporated in a consistent physical scheme, and invite workers pursuing this direction to address what, to us, seems like a neglected pathology. In conclusion, then, we think it fair to say that interpreting the Z's as spacetime coordinate operators of a particle makes it improbable for the α_2 deformation to have the physical applications proposed in the literature. On the other hand, our identification of the Z's with the moment operators leads to the conclusion that $\mathcal{G}_{\mathrm{QR}}$, the standard, experimentally tested, quantum relativistic algebra in which, in particular, the spacetime coordinates commute, is given, in the case of a massive spinless particle, by $\alpha_2 = q$, with the experiment fixing the value $q = \hbar$. If the interpretation advocated above is correct, then, a look at Fig. 2 shows that the only deformations left to explore are those generated by $\pm\chi$, leading to the future or the elsewhere, respectively, both introducing non-commutativity of the momenta.

The α_3 deformation signals a more radical departure from $\mathcal{G}_{\mathrm{QR}}$, so much so that, in Ref. 27, it is practically discarded as unphysical. Reference 19, on the other hand, treats it on an equal footing and observes that, with the Z's as positions, α_3 is dimensionless, so that $\hbar\alpha_3$ is a new fundamental constant with dimensions of action. When the Z's are taken as moments, α_3 acquires dimensions of mass. In either case, the physical implications of the deformation are somewhat obscure and deserve further study.

5. Concluding Remarks

We have pursued in this paper the stability point of view to its ultimate consequences. Our systematic algebraic analysis has recovered previous results, establishing their uniqueness, and shedding light along the way on various technical issues, in particular, the interrelations among the deformations found. A fundamental departure from the established lore has been our identification of the Z_μ generators with the moment operators of a (massive, spinless) particle, having concluded that the position operators lack the essential property of primitiveness, necessary for all Lie algebra generators.

We think that a number of questions raised here deserve further study. First, we would like to extend the concept of the moment operators to the case of particles with spin, and/or zero mass — this should be subsequently generalized to the fully deformed case. Second, representation theoretical aspects of the problem should be examined, in particular, a Wigner-type classification should be carried through for the deformed algebras. It would also be of interest to develop some degree of intuition regarding the deformed kinematics, e.g., by clarifying the coexistence of the Lorentz contraction with an invariant length scale. Further ahead, one can wonder about the form quantum field theory would take in the spacetimes we have been considering, and whether the invariant scales introduced by the deformations do indeed provide natural cutoffs. Still further on the horizon, a supersymmetric version of this work, including this section's musings, can be envisioned.

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Appendix A. Computing $H^2(\mathcal{G}_{\rm CR})$

The complexity of the calculation of the second cohomology group of an algebra grows rapidly with its dimension. When dealing with a 15-dimensional algebra, like $\mathcal{G}_{\rm CR}$ in the case at hand, the prospect of carrying out the analysis manually becomes somewhat unattractive. Luckily, some MATHEMATICA code we wrote deals with the problem within minutes — we give here some details of the calculations. The algorithm we used was the following:

(1) Consider the most general 1-cochain ϕ,

$$\phi = {\phi_A}^B \Pi^A \otimes T_B \,, \tag{A.1}$$

with ${\phi_A}^B$ arbitrary real constants (a sum of $15^2 = 225$ terms). Obtain the most general 2-coboundary ψ by setting $\psi = \nabla\phi$. This produces a sum of 1008 terms, each corresponding to a non-zero component of ψ.

(2) Consider the most general 2-cochain χ,

$$\chi = {\chi_{AB}}^C \Pi^A \Pi^B \otimes T_C \,, \tag{A.2}$$

with ${\chi_{AB}}^C$ arbitrary real constants (a sum of $15\binom{15}{2} = 1575$ terms). Require that it be a 2-cocycle by setting $\nabla\chi = 0$. This results in a system of 5672 linear homogeneous equations in the above 1575 ${\chi_{AB}}^C$'s, which is solved for some of them in terms of the rest — call the latter c_i. Effecting these substitutions in

χ, one obtains the most general 2-cocycle $\tilde{\chi} \equiv \sum_i c_i \chi_i$ with *arbitrary* c_i. As a result, each of the χ_i in the sum is by itself a 2-cocycle — there are 221 of them in our case.

(3) Examine which of the χ_i's are non-trivial, i.e., check if the equations $\chi_i = \psi$ have a solution for the ${\phi_A}^B$ that appear in ψ. For each χ_i, this produces a system of 1575 equations. If a solution exists, the 2-cocycle in question is trivial, i.e., a 2-coboundary. For the problem at hand, 5 out of the 211 χ_i turn out to be non-trivial.

(4) Check whether the non-trivial cocycles obtained correspond to *independent* generators of $H^2(\mathcal{G}_{\mathrm{CR}})$. Do this by setting an arbitrary linear combination of the 2-cocycles equal to the general 2-coboundary. If a solution for the ${\phi_A}^B$ exists, discard one of the cocycles that enter in the linear combination and repeat the test for the remaining ones, until no solution exists. For the case at hand, no linear dependence was found, arriving thus at the final result, Eq. (50).

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